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Asymptotic analysis of nonlinear dynamics of simply supported cylindrical shells

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Abstract Donnell equations are used to simulate free nonlinear oscillations of cylindrical shells with imperfections. The expansion, which consists of two conjugate modes and axisymmetric one, is used to analyze shell oscillations. Amplitudes of the axisymmetric motions are assumed significantly smaller, than the conjugate modes amplitudes. Nonlinear normal vibrations mode, which is determined by shell imperfections, is analyzed. The stability and bifurcations of this mode are studied by the multiple scales method. It is discovered that stable quasiperiodic motions appear at the bifurcations points.

The forced oscillations of circular cylindrical shells in the case of two internal resonances and the principle resonance are analyzed too. The multiple scales method is used to obtain the system of six modulation equations. The method for stability analysis of standing waves is suggested. The continuation algorithm is used to analyze fixed points of the system of the modulation equations.

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1. Introduction

Cylindrical shells are widely used in aerospace, mechanical and civil engineering. Therefore, many efforts were made to study nonlinear oscillations of cylindrical shells. One of the important problems in nonlinear dynamics of shells is to choice mode expansions of oscillations. On the one hand, the mode expansion must approximate adequately the shell oscillations, but on the other hand, this expansion must contain the minimum number of degrees-of-freedom. Evensen and Kubenko [1–3] used the results of the linear analysis and experimental data of nonlinear vibrations to choice the best mode expansions. Dowell and Ventress [4] show that mode expansion must satisfy the periodicity condition of circumference displacements and geometrical boundary conditions. Evensen [1–2] obtained that the frequency response is hard without the axisymmetrical part of mode expansion, but it is soft if these motions are considered. The soft frequency response is in a good agreement with the experimental data. Moreover, Evensen show that an inextensibility condition of the middle surface must be taken into account to obtain the soft frequency response. One and two modes approximations of plates and shells oscillations are considered by Vol'mir [5]. Atluri [6] used perturbation techniques to study dynamical system, which describe three mode expansion of the shell oscillations. Hsu [7] studied parametric oscillations of simply supported and clamped- free cylindrical shells. Koval [8] considered the parametric oscillations of cylindrical shells taking into account both longitudinal and bending motions. He showed that the first eigenfrequency corresponds to the significant bending and small longitudinal motions. Lockhart [9] analyzed the dynamics of geometrically nonlinear cylindrical shell with imperfections under the action of impulse longitudinal force. He presented the dynamic response in the form of power series with respect to small parameter. It is shown in the papers [3, 10] that imperfections and nonlinear inertia terms affect significantly on nonlinear oscillations of cylindrical shell. Nayfeh and Raouf [11] used McIvor's model to study forced oscillations of infinitely long cylindrical shell. They considered the saturation phenomenon, when energy of the axisymmetric mode is pumped over the asymmetrical motions. Koval'shuk and Podshasov [12] derived the quasilinear system of ODEs to describe the traveling waves in cylindrical shells. They used the average method to study the main resonance. Raouf and Nayfeh [13] considered the forced oscillations of the infinity long cylindrical shell close to the principal resonance. They derived the system of modulation equations and studied periodic solutions, their bifurcations and chaotic motions. Detailed review of experimental results of cylindrical shells nonlinear oscillations is presented in the paper [14], where the experimental data of bifurcations of traveling waves, the regions of beating and subharmonic oscillations are reported. Manevich [15] considered free oscillations of rings taking into account an energy exchange between conjugate modes. Two nonlinear partial differential equations with respect to radial and circumferential displacements are used to analyze the shell oscillations by Chin and Nayfeh [16]. They derived the system of ODEs with inertial and cubic nonlinearities. An orthotropic, geometrically nonlinear cylindrical shell is considered in the paper [17], where it is assumed, that the amplitudes of axisymmetric breathing are less than the amplitudes of asymmetrical modes. Ladyguina and Manevich [18] considered the interaction of conjugate modes by the multiple scales method. Kubenko and Koval'shuk [19] obtained that the traveling waves take place in the narrow resonance domain and the standing waves occur outside this region. Moreover, they obtained that the frequency response of the parametrically excited shell is soft-hard in the region of the main parametric resonance. Four modes expansion of the shell nonlinear oscillations is considered by Amabili, Pellicano and Vakakis [20]. They use the normal forms method to study the four-degree-of-freedom system. Based on experimental data, Amabili, Pellegrini and Pellicano [21] are made the conclusion, that the soft frequency response of short shells has larger puling of frequency, than the long one. Pellicano, Amabili, and Paidoussis [22] studied the effect of shell parameters on hardening – softening behavior of the frequency responses. The interaction of four modes is considered to study free oscillations of cylindrical shells in the paper [23]. Pellicano and Avramov [24] considered a nonlinear dynamics of a circular cylindrical shell carrying a rigid disk.

Recently a lot of papers devoted to many mode models of nonlinear oscillations of cylindrical shells appear. These models are studied numerically. Another approach for nonlinear oscillations of cylindrical shells based on asymptotic analysis is considered in this paper. Discrete models with small number of DOF developed in [1, 3] are used to study nonlinear oscillations analytically here. The results obtained by these models are in good agreement with the experimental data. The aim of this paper is to analyze nonlinear normal modes of free oscillations of cylindrical shells and to investigate forced oscillations in the case of two internal resonances and the principal resonance.

In this paper the method of nonlinear normal vibrations modes is used to study free oscillations of cylindrical shell. It is shown that a single nonlinear normal mode is mainly determined by imperfections of shell. Stability and bifurcations of this mode are investigated by the multiple scales method.

The forced oscillations of cylindrical shells in the case of two internal resonances are studied analytically by the multiple scales method. The hard frequency responses of standing and traveling waves, which are explained by addition internal resonance of the discrete model of shell, are obtained. The new method for stability analysis of standing waves is suggested in this paper too.

2. Discrete models of oscillations

The nonlinear oscillations of the cylindrical shells (Fig. 1) with imperfections can be described by the

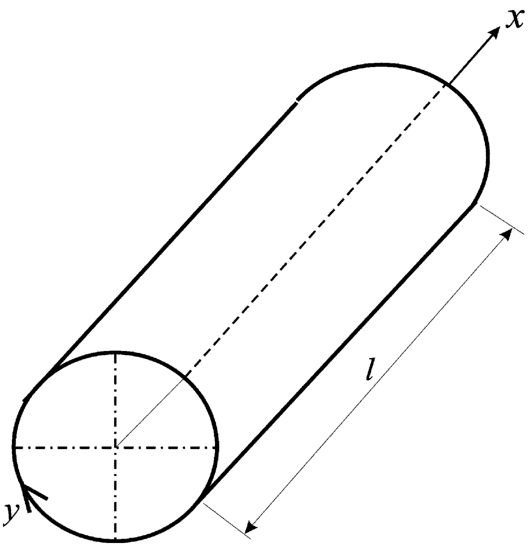


Fig. 1 Cylindrical shell principal model

Donnell Equations [25, 26]:

$$\frac{D}{h}\nabla^4 w_1 = L(w_1 + w_0, \Phi) + \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} - \rho \frac{\partial^2 w_1}{\partial t^2} + \frac{q}{h}; \quad (1)$$

$$\frac{1}{E}\nabla^4 \Phi = -\frac{1}{2}L(w_1 + 2w_0, w_1) - \frac{1}{R}\frac{\partial^2 w_1}{\partial x^2};$$

$$L(A, B) = \frac{\partial^2 A}{\partial x^2}\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 A}{\partial y^2}\frac{\partial^2 B}{\partial x^2} - 2\frac{\partial^2 A}{\partial x \partial y}\frac{\partial^2 B}{\partial x \partial y}, \quad (2)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$; E is the Young modulus; ν is the Poisson ratio; ρ is the shell density; R, h are radius and thickness of the shell; x, y are the longitudinal and circumference coordinates (Fig. 1); w_1 is dynamical deflections; Φ is the in-plane Airy stress function; $w_0(x, y)$ is initial imperfections; q is external excitation. If free oscillations are considered, then $q = 0$.

The following expansion of shell oscillations is used [1]:

$$w_1 = f_1(t)\cos sy \sin rx + f_2(t)\sin sy \sin rx + f_3(t)\sin^2 rx, \quad (3)$$

where $s = n/R$; $r = m\pi/l$, n is the number of waves in the circumference direction; m is the number of half-wave in the longitudinal direction; l is a length of the shell. The function $\sin^2 rx$ of the expansion (3) does not satisfy the boundary condition for moment at $x = 0$ and $x = l$, but as shown in [1, 2] only the boundary conditions for w_1 at $x = 0$ and $x = l$ can be satisfied.

2.1. Model of free oscillations of shells with imperfections

It is assumed that the length of the middle surface transverse section is constant during oscillations [1]:

$$\int_0^{2\pi R} \varepsilon_{22}\, dy = \int_0^{2\pi R} \left[\frac{\partial V}{\partial y} - \frac{w}{R} + \frac{R}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] dy = 0, \quad (4)$$

where V is circumference displacements. The following equation is derived from (4): $f_3 = (f_1^2 + f_2^2)n^2/(4R)$. Imperfections are considered in the following form:

$$w_0 = f_{10}\cos sy \sin rx + f_{20}\sin sy \sin rx, \quad (5)$$

where f_{10}, f_{20} are constant values. The Equations (3) (5) are substituted into (2). Then the Equation (2) is solved analytically and the results are substituted into (1). Galerkin method is used to Equation (1). As a result the following dynamical system is derived [3]:

$$\ddot{f}_1 + \omega_1^2 f_1 + \gamma \dot{f}_2 + 2\chi f_1(\dot{f}_1^2 + f_1\ddot{f}_1 + \dot{f}_2^2 + f_2\ddot{f}_2) + \gamma_1 f_1(f_1^2 + f_2^2) + g f_1(f_1^2 + f_2^2)^2 + \alpha_1 f_1 f_2 + \alpha_2 f_1^2 + \alpha_3 f_2^2 = \omega_0^2 f_{10}; \quad (6)$$

$$\ddot{f}_2 + \omega_2^2 f_2 + \gamma \dot{f}_1 + 2\chi f_2(\dot{f}_1^2 + f_1\ddot{f}_1 + \dot{f}_2^2 + f_2\ddot{f}_2) + \gamma_1 f_2(f_1^2 + f_2^2) + g f_2(f_1^2 + f_2^2)^2 + \beta_3 f_1 f_2 + \beta_1 f_1^2 + \beta_2 f_2^2 = \omega_0^2 f_{20}, \quad (7)$$

where parameters $\chi, \gamma, \omega_1^2, \omega_2^2, \gamma_1, g, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are presented in Appendix. Let us introduce the dimensionless variables and parameters:

$$t_* = \omega_0 t; \quad x_* = f_1 h^{-1}; \quad y_* = f_2 h^{-1};$$
$$\omega_1^* = \omega_1\omega_0^{-1}; \quad \omega_2^* = \omega_2\omega_0^{-1}; \quad \varepsilon^2\gamma^* = \gamma\omega_0^{-2};$$
$$\chi^* = \chi h^2; \quad \gamma_1^* = \gamma_1\omega_0^{-2}h^2; \quad g^* = g\omega_0^{-2}h^4;$$
$$\varepsilon\alpha_1^* = \alpha_1\omega_0^{-2}h; \quad \varepsilon\alpha_2^* = \alpha_2\omega_0^{-2}h; \quad \varepsilon\alpha_3^* = \alpha_3\omega_0^{-2}h;$$
$$f_{10}^* = f_{10}h^{-1}; \quad f_{20}^* = f_{20}h^{-1}; \quad \varepsilon\beta_1^* = \beta_1 h\omega_0^{-2};$$
$$\varepsilon\beta_2^* = \beta_2 h\omega_0^{-2}; \quad \varepsilon\beta_3^* = \beta_3 h\omega_0^{-2}, \quad (8)$$

where $\varepsilon \ll 1$ is a small parameter. The system (6, 7) is written with respect to dimensionless variables and parameters dropping the symbol "*" in the notations (8):

$$\begin{aligned}
&\ddot{x} + \omega_1^2 x + \varepsilon^2 \gamma y + 2\chi x(\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}) \\
&\quad + \gamma_1 x(x^2 + y^2) + g x(x^2 + y^2)^2 + \varepsilon\alpha_1 xy + \varepsilon\alpha_2 x^2 \\
&\quad + \varepsilon\alpha_3 y^2 = f_{10}; \\
&\ddot{y} + \omega_2^2 y + \varepsilon^2 \gamma x + 2\chi y(\dot{y}^2 + y\ddot{y} + \dot{x}^2 + x\ddot{x}) \\
&\quad + \gamma_1 y(x^2 + y^2) + g y(x^2 + y^2)^2 + \varepsilon\beta_1 x^2 + \varepsilon\beta_2 y^2 \\
&\quad + \varepsilon\beta_3 xy = f_{20}.
\end{aligned} \tag{9}$$

the parameter ε indicates only the small terms in the system (9).

2.2. Model of Forced Oscillations

It is assumed that the external excitation corresponds to the mode $\cos sy \sin rx$:

$$q = E_1 \cos sy \sin rx \cos \Omega t. \tag{10}$$

The shell deflection is considered in the form (3). The expansion (3) is substituted into (2). Then the solution of Equation (2) is

$$\begin{aligned}
\Phi = {}& \Phi_0 \cos 2rx + \Phi_1 \sin rx \cos sy + \Phi_2 \sin rx \sin sy \\
& + \Phi_3 \cos 2sy + \Phi_4 \sin 2sy + \Phi_5 \sin 3rx \cos sy \\
& + \Phi_6 \sin 3rx \sin sy - 0.5Kx^2,
\end{aligned} \tag{11}$$

where $\Phi_0, \Phi_1, \ldots, \Phi_6$ depends on the parameters of shells. The term $-0.5Kx^2$ of (11) describes the membrane stresses of the middle surface. The periodicity condition of circumference displacements is used to determine the constant K:

$$\int_0^{2\pi R} \frac{\partial V}{\partial y} dy = \int_0^{2\pi R} \left\{ \frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial y^2} - \frac{E}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{Ew}{R} \right\} dy = 0. \tag{12}$$

Then it is derived:

$$K = -\frac{Es^2}{8}\left(f_1^2 + f_2^2 - \frac{4f_3}{Rs^2}\right). \tag{13}$$

The in-plane stress function (11) is substituted into Equation (1) and the Galerkin method is used. As a result, the following system of ODEs is obtained:

$$\begin{aligned}
&\ddot{f}_1 + \omega^2 f_1 + \delta_1 \dot{f}_1 + \gamma_1 f_1 f_3 + \gamma_2 f_1 (f_1^2 + f_2^2) \\
&\quad + \gamma_3 f_1 f_3^2 = \alpha_1 \cos \Omega t; \\
&\ddot{f}_2 + \omega^2 f_2 + \delta_2 \dot{f}_2 \\
&\quad + \gamma_1 f_2 f_3 + \gamma_2 f_2 (f_1^2 + f_2^2) + \gamma_3 f_2 f_3^2 = 0; \\
&\ddot{f}_3 + \omega_3^2 f_3 + \delta_3 \dot{f}_3 + \frac{\gamma_1}{3}(f_1^2 + f_2^2) \\
&\quad + \frac{2}{3}\gamma_3 f_3 (f_1^2 + f_2^2) = 0;
\end{aligned} \tag{14}$$

where $\alpha_1 = E_1/\rho h$; $\delta_1, \delta_2, \delta_3$ are damping coefficients; Ω is frequency of excitation, the parameters $\omega^2, \omega_3^2, \gamma_1, \gamma_2, \gamma_3$are presented in Appendix.

The small parameter μ is introduced: $\mu = h/R \ll 1$. The dimensionless general coordinates $x_i = f_i h^{-1} (i = 1, 2, 3)$ and the dimensionless time $\tau = \omega t$ are used. Then the system (14) has the following dimensionless form:

$$\begin{aligned}
&\ddot{x}_1 + x_1 + \mu\left[\bar{\delta}_1 \dot{x}_1 + \bar{\gamma}_1 x_1 x_3 + \bar{\gamma}_2 x_1 (x_1^2 + x_2^2) \right. \\
&\quad \left. + \bar{\gamma}_3 x_1 x_3^2\right] = \mu \bar{\alpha}_1 \cos p\tau; \\
&\ddot{x}_2 + x_2 + \mu\left[\bar{\delta}_2 \dot{x}_2 \right. \\
&\quad \left. + \bar{\gamma}_1 x_2 x_3 + \bar{\gamma}_2 x_2 (x_1^2 + x_2^2) + \bar{\gamma}_3 x_2 x_3^2\right] = 0; \\
&\ddot{x}_3 + \bar{\omega}_3^2 x_3 + \mu\left[\bar{\delta}_3 \dot{x}_3 + \frac{\bar{\gamma}_1}{3}(x_1^2 + x_2^2) \right. \\
&\quad \left. + \frac{2}{3}\bar{\gamma}_3 x_3 (x_1^2 + x_2^2)\right] = 0,
\end{aligned} \tag{15}$$

where $\bar{\omega}_3 = \omega_3 \omega^{-1}$; $p = \Omega\omega^{-1}$; $\mu\bar{\gamma}_1 = h^2 \gamma_1 \omega^{-2}$; $\mu\bar{\gamma}_2 = h^2 \gamma_2 \omega^{-2}$; $\mu\bar{\gamma}_3 = h^2 \gamma_3 \omega^{-2}$; $\mu\bar{\alpha}_1 = \alpha_1 \omega^{-2} h$; $\mu\bar{\delta}_1 = \delta_1 \omega^{-1}$; $\mu\bar{\delta}_2 = \delta_2 \omega^{-1}$; $\mu\bar{\delta}_3 = \delta_3 \omega^{-1}$; the parameters $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$ are presented in Appendix.

The internal resonance 1:1 is always observed in the system (15). Moreover, the additional internal resonance between the frequencies $\bar{\omega}_1 = 1$ and $\bar{\omega}_3$ can take place in the system (15). The system (15) with such two resonances is considered in this paper. Note that the dynamical system (15) and the model (9) have essential distinctions. If the transversal section of the middle surface is considered inextensible, the general coordinate f_3 is the function of f_2 and f_1:

$f_3 = n^2(f_1^2 + f_2^2)/(4R)$. That is f_3 has the smaller order than f_1 and f_2: $f_3 = O(f_1^2, f_2^2)$. In this case the shell model (9) has two-degree-of-freedom. As shown in [3] this model has good agreement with the experimental data. However, this two degree-of-freedom shell model does not describe the internal resonances $\bar{\omega}_3 \approx 2$ and $\bar{\omega}_3 \approx 1$. As it will be shown in this paper, these internal resonances are observed in the first approximation by μ of the system (15). In this case the variable f_3 can not be small and oscillation energy of two conjugate modes pumped into the axisymmetric motions. This effect can not be observed in two-degree-of-freedom system (9). Therefore, the model (9) can not be obtained from the system (15) by using the relation $\bar{\alpha}_1 = 0$. As shown in [3], the results of analytical analysis of the three-degree-of-freedom system (15) in the vicinity of the main resonance without the internal resonances $\bar{\omega}_3 \approx 2$ and $\bar{\omega}_3 \approx 1$ are close to the experimental data. The detailed analysis of the forced oscillations with addition internal resonance $\bar{\omega}_3 \approx 2$ is performed in this paper.

3. Nonlinear normal modes of shells

The nonlinear normal modes (NNMs), which are the generalization of the normal modes of linear systems, are considered in [27, 28]. The method of NNMs is an effective tool to study engineering problems. For example, this method was used for the problem of vibrations absorption [29 – 31]. Here NNMs of cylindrical shell is considered. If the shell imperfections are equal to zero, all NNMs trajectories in the configuration space of the system (9) are rectilinear and they have the following form:

$$y = kx. \tag{16}$$

To study the rectilinear approximation (16) of NNMs, the following approach is used. The Equation (16) is substituted into the system (9). The following equations are derived:

$$\begin{aligned}
&\ddot{x} + 2\chi(1+k^2)x(\dot{x}^2 + x\ddot{x}) = -\gamma_1(1+k^2)x^3\\
&\quad -(\alpha_1 k + \alpha_2 + \alpha_3 k^2)x^2 + \omega_0^2 f_{10} - \left(\omega_1^2 + \gamma k\right)x;\\
&\ddot{x} + 2\chi(1+k^2)x(\dot{x}^2 + x\ddot{x}) = -\gamma_1(1+k^2)x^3\\
&\quad -(\beta_1 k^{-1} + \beta_2 k + \beta_3)x^2 + k^{-1}\omega_0^2 f_{20}\\
&-\left(\omega_2^2 + \gamma k^{-1}\right)x,
\end{aligned} \tag{17}$$

where $x(t)$ describes the motion on the NNM (16). Then the coordinate x is replaced by its amplitude value X. Now the right-hand parts of (17) are equated and the following algebraic equation as a condition of compatibility is derived:

$$\begin{aligned}
&X\left[k\left(\Omega_1^2 - \Omega_2^2\right) + \gamma(k^2-1)\right] + X^2\left[k^3\frac{\beta_3}{2} + k^2(2\beta_1\right.\\
&\qquad \left. -\beta_2) + k(\alpha_2 - \beta_3) - \beta_1\right] = \omega_0^2(kf_{10} - f_{20}),
\end{aligned} \tag{18}$$

where $\alpha_1 = 2\beta_1; \beta_3 = 2\alpha_3; \Omega_{1,2}^2 = \omega_0^2 - \omega_{1,2}^2$. The Equation (18) is cubic with respect to k. This cubic equation is solved numerically with the different values of f_{10}, f_{20}, X. The obtained NNMs are verified by the direct numerical integration of the system (9). As a result of the calculations, the following conclusion can be made. In the wide range of the system parameters, only the single NNM (16) exists with k close to

$$k = f_{20}/f_{10}. \tag{19}$$

Figure 2 shows the results of direct numerical simulations of NNM (19, 16). The following shell parameters are used: $n = 7$; $R = 0.2\,\text{m}$; $f_{20} = f_{10}$; $L = 1.4$; $f_{10} = 0.2 \times 10^{-3}\,\text{m}$; $h = 0.5 \times 10^{-3}\,\text{m}$; $X = 2\,\text{h}$; $E = 2 \times 10^{11}\,\text{Pa}$; $\rho = 7800\,\text{kg/m}^3$; $\nu = 0.3$. The numerical results are in agreement with the NNM (19, 16). Fig. 3 shows the trajectory in the configuration space, which was obtained by direct numerical integration of the system (9) with the same parameters as at the Fig. 2 and with the coefficient $k = -1.58$ of the Equation (16). As will be shown below, this trajectory (Fig. 3) corresponds to a case of the NNM instability.

To analyze the NNMs with curvilinear trajectories in the configuration space, the equation of motion in configuration space will be derived. The variable x of the system (9) is used as independent variable instead of t. Then the solution of the system (9) is considered in the following form: $y(x)$. The following formulae are used:

$$\frac{d}{dt} = \dot{x}\frac{d}{dx}, \quad \frac{d^2}{dt^2} = \dot{x}^2\frac{d^2}{dx^2} + \ddot{x}\frac{d}{dx}. \tag{20}$$

The energy integral of the system (9) has the following form:

$$T + \Pi = H, \tag{21}$$

Fig. 2 The nonlinear normal vibration mode trajectory for the following shell parameters: $h = 0.5 \times 10^{-3}$ m; $X = 2$ h; $E = 2 \times 10^{11}$ Pa; $\rho = 7800\,\text{kg/m}^3$; $\nu = 0.3$; $f_{10} = 0.2 \times 10^{-3}$ m; $k = 1$; $L = 1.4$. $n = 7$; $R = 0.2$ m; $f_{20} = f_{10}$

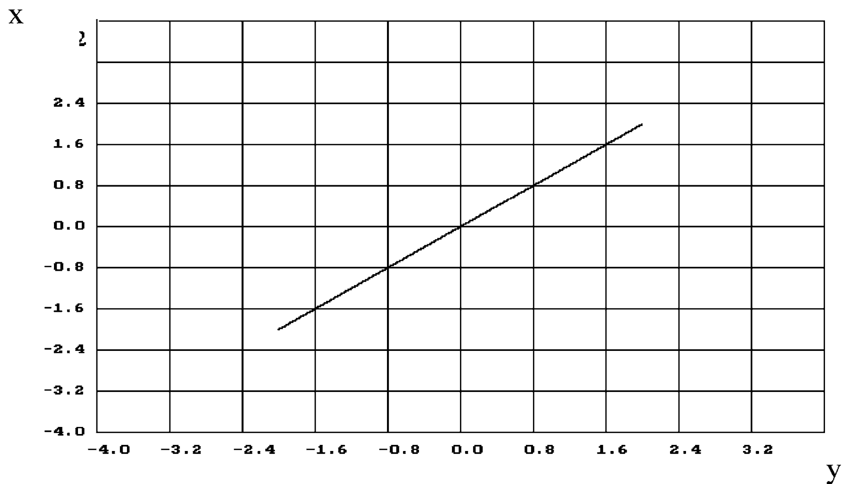
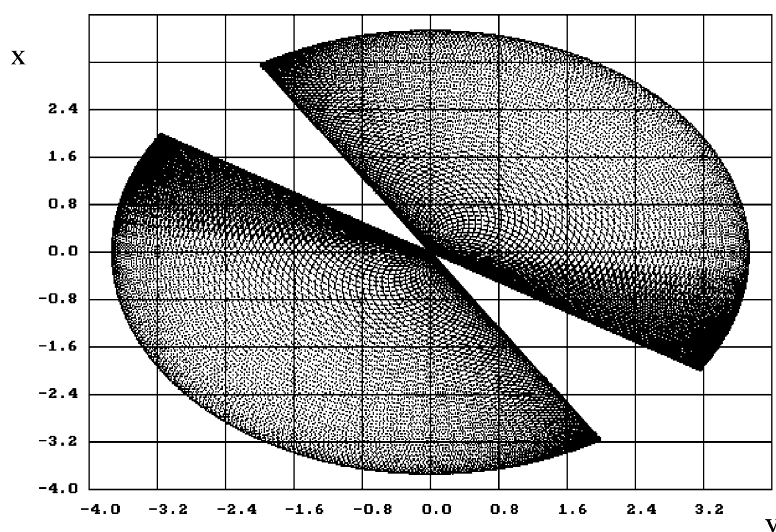


Fig. 3 The trajectory in the configuration space for the following shell parameters: $n = 7$; $k = -1.58$; $h = 0.5 \times 10^{-3}$ m; $X = 2$ h; $E = 2 \times 10^{11}$ Pa; $\rho = 7800\,\text{kg/m}^3$; $\nu = 0.3$; $f_{10} = 0.2 \times 10^{-3}$ m; $L = 1.4$; $R = 0.2$ m



$$T = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + T_1,\ T_1 = \chi(\dot{x}^2 x^2 + 2xy\dot{x}\dot{y} + \dot{y}^2 y^2);$$

$$\Pi = \omega_1^2 \frac{x^2}{2} + \omega_2^2 \frac{y^2}{2} + \gamma xy + \gamma_1 \left(\frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2 y^2}{2}\right) + \frac{g}{6}(x^2 + y^2)^3 + \alpha_1 \frac{x^2 y}{2} + \alpha_2 \frac{x^3}{3} + \beta_2 \frac{y^3}{3} + \beta_3 \frac{xy^2}{2} - \omega_0^2 f_{10} x - \omega_0^2 f_{20} y,$$

where H is a constant value of energy. It is derived from (21):

$$\dot{x}^2 = \frac{2(H - \Pi)}{1 + y'^2 + 2T_1}. \tag{22}$$

Here the prime means a derivation by x. Using the formulas (9, 21, 22), the equations of motions in configuration space are obtained as

$$\frac{2(H - \Pi)}{1 + y'^2 + 2T_1} y'' - (2\chi x P + \Pi'_x) y' + 2\chi y P = -\Pi'_y; \tag{23}$$

$$P = \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}. \tag{24}$$

After some algebra with the relations (20–22), the following formula is obtained:

$$P = \dot{x}^2(1 + y'^2 + yy'') + \ddot{x}(x + yy').$$

Using the Equations (20, 22), it is obtained:

$$P=\frac{2(H-\Pi)(1+y'^2+yy'')}{(1+y'^2+2T_1)[1+2\chi x(x+yy')]}-\frac{(x+yy')}{1+2\chi x(x+yy')}\Pi'_x. \quad (25)$$

The Equation (25) is substituted into (23). Then the equation of motions in the configuration space is derived

$$\frac{2(H-\Pi)}{1+y'^2+2T_1}\left[y''+2\chi(y-xy')\frac{1+y'^2+yy''}{1+2\chi x(x+yy')}\right]-\Pi_x\left[y'+\frac{(x+yy')2\chi(y-xy')}{1+2\chi x(x+yy')}\right]=-\Pi_y. \quad (26)$$

The Equation (26) has singularities at the maximal isoenergetic surface $\Pi=H$. The analytical continuation of trajectories up to this surface is possible due to the boundary condition [27]:

$$\left\{-\Pi_x\left[y'+\frac{(x+yy')2\chi(y-xy')}{1+2\chi x(x+yy')}\right]=-\Pi_y\right\}\Bigg|_{\Pi(X,y(X))=H}. \quad (27)$$

This boundary condition is obtained from (26) assuming that $\Pi=H$. More deep consideration of boundary conditions for NNMs the reader can find in the book [27].

Let us assume that imperfections f_{10}, f_{20} are small. Then the addition small parameter $\varepsilon_1 \ll 1$ is introduced and the following estimation is used: $(f_{10}, f_{20})=O(\varepsilon_1)$. The orders of the dynamical system (9) coefficients with respect to the imperfections are the following:

$$\alpha_1=O(f_{20});\quad (\alpha_2,\alpha_3)=O(f_{10});\quad \beta_3=O(f_{10});\quad \gamma=O(f_{10}f_{20});\quad (\beta_1,\beta_2)=O(f_{20}).$$

Then the NNM trajectory is represented by the asymptotic expansion:

$$y=kx+\varepsilon_1 y_1(x)+\cdots \quad (28)$$

As $\varepsilon_1 \ll 1$, the NNM (28) is close to rectilinear one. The expansion (28) is substituted into the Equation (26) and the terms of order $O(\varepsilon_1)$ are equated. As a result it is obtained the next equation of the first order with respect to ε_1:

$$2y_1''\frac{H-\Pi^0}{1+k^2+2T_1^0}+4\chi\frac{H-\Pi^0}{1+k^2+2T_1^0}\frac{1+k^2}{1+2\chi(1+k^2)x^2}(y_1-xy_1')-\Pi_x^0\frac{(1+k^2)2\chi x^2(y_1-xy_1')}{1+2\chi(1+k^2)x^2}=G+\Pi_x^1 k-\Pi_y^1, \quad (29)$$

where $\Pi^0=\Pi|_{y=kx,\varepsilon=0}$; $G=\Pi_{xy}^0 k y_1+\Pi_x^0 y_1'-\Pi_{yy}^0 y$. The Equation (28) is substituted into the boundary condition (27). Then the boundary condition of order $O(\varepsilon_1)$ is derived:

$$\left[\Pi_x^0\frac{(1+k^2)2\chi x^2(y_1-xy_1')}{1+2\chi(1+k^2)x^2}+\Pi_x^1 k-\Pi_y^1\right]\Bigg|_{H=\Pi}=0. \quad (30)$$

The solution of Equation (29) can be presented as power series [27]:

$$y_1=a_0+a_1x+a_2x^2+\ldots \quad (31)$$

The series (31) is substituted into the Equation (29) and the system of linear algebraic equations with respect to $a_1, a_2, \ldots$ is derived. The series (31) is substituted into the boundary condition (30) too and the additional single linear algebraic equation is obtained. The obtained system of linear algebraic equations is not presented here. As the internal resonance 1:1 is observed in the system (9) without the initial imperfections, the determinant of the system of linear algebraic equations is close to zero. Note, that the parameters $\gamma, \alpha_i, \beta_i$ of the system (9) are smaller by a factor of $30 \div 100$, than f_{10}, f_{20}. As a result, it can obtain that the solving condition of the system of linear algebraic equation is the following:

$$\left[\omega_0^2(f_{10}/\varepsilon_1)x-\omega_0^2(f_{20}/\varepsilon_1)y\right]\Big|_{y=kx}=0.$$

Then the value $k=f_{10}/f_{20}$ is obtained from the last equation. Thus, only the single nonlinear normal mode $y\approx f_{10}x/f_{20}$ close to rectilinear one exists in the case of small imperfections and this NNM is mainly determined by these imperfections. Of course, others NNMs and others kinds of periodic and nonperiodic motions can exist in the system (9).

4. Bifurcations and stability of nonlinear normal mode

In this section the stability and bifurcations of the NNM (28, 19) is studied. It is important, that if $\varepsilon = 0$, the system (9) is essentially nonlinear. The system (9) is rewritten with respect to polar coordinates $r = \sqrt{x^2 + y^2}; \tan\theta = \frac{y}{x}$:

$$\begin{aligned} &\ddot{\theta}_1 r^2 + 2r\dot{r}\dot{\theta}_1 + \varepsilon r^2[\varepsilon\gamma\cos 2(\theta_1 + \theta_*) \\ &\quad +\varepsilon\Omega\cos 2(\theta_1 + \theta_*)] + F_0(r - \varepsilon S r^3)\sin\theta_1 = 0; \\ &(1 + 2\chi r^2)\ddot{r} + 2\chi r\dot{r}^2 - \dot{\theta}_1^2 r + r + \gamma_1 r^3 + g r^5 \\ &\quad +\varepsilon r[\varepsilon\gamma\sin 2(\theta_1 + \theta_*) - \varepsilon\Omega\cos 2(\theta_1 + \theta_*)] \\ &\quad -F_0(1 - \varepsilon S 3 r^2)\cos\theta_1 = 0, \end{aligned} \tag{32}$$

where $\theta_1 = \theta - \theta_*; \cos\theta_* = f_{10}/F_0; \sin\theta_* = f_{20}/F_0$; $F_0 = \sqrt{f_{10}^2 + f_{20}^2}$; $\varepsilon^2\Omega = \frac{Er^4}{16\rho\omega_0^2}(f_{10}^2 - f_{20}^2)$; $\varepsilon S = h\tilde{\sigma}(4\omega_0^2); \tilde{\sigma} = Er^4 s^4 \rho^{-1}(s^2 + r^2)^{-2}$. Only two parameters of the system (32) $\varepsilon^2\Omega$ and $\varepsilon^2\gamma$ have the second order with respect to ε. The others parameters have the orders $O(1)$ and $O(\varepsilon)$. Therefore, in the future treatments the terms with parameters $\varepsilon^2\Omega, \varepsilon^2\gamma$ are not consider.

The equation $\theta_1 = 0$ determines the NNM of the system (32), which has the form: $y = f_{20}x/f_{10}$ in the Cartesian coordinates. This NNM is considered in the previous section.

Let us determine the fixed point of the system (32), which satisfies the equations: $\dot{\theta}_1 = \dot{r} = 0$. This fixed point is

$$\begin{aligned} &r_\Sigma = \bar{r}_0 - \varepsilon\frac{\bar{r}_0 D + 3F_0 S\bar{r}_0^2}{1 + 3\gamma_1\bar{r}_0^2 + 5g\bar{r}_0^4} + O(\varepsilon^2); \\ &\theta_1 = 0 + O(\varepsilon^2), \end{aligned} \tag{33}$$

where $D = \gamma\sin(2\theta_*) - \Omega\cos(2\theta_*)$. The parameter $\bar{r}_0$ satisfies the nonlinear equations:

$$\psi(\bar{r}_0) = F_0; \quad \psi(r) = r + \gamma_1 r^3 + g r^5. \tag{34}$$

Now stability and bifurcations of the NNM with small amplitudes are considered close to the fixed point (33). The following change of variables is used:

$$r = r_\Sigma + \varepsilon R(t); \theta_1 = \varepsilon\phi(t), \tag{35}$$

The variables $R(t), \phi(t)$ satisfy the following equations:

$$\begin{aligned} &\ddot{R} + \alpha_2^2 R \\ &\quad = \frac{\varepsilon}{1 + 2\bar{r}_0^2\chi}\left[GR^2 + C + \bar{r}_0^- 2\bar{r}_0\chi\dot{R}^2 - \frac{F_0^2}{2}\right]; \end{aligned} \tag{36}$$

$$\ddot{\phi} + \alpha_1^2\phi = \varepsilon\left(\frac{F_0}{\bar{r}_0^2}R\phi - \frac{2}{\bar{r}_0}\dot{R}\dot{\phi}\right), \tag{37}$$

where

$$G = \frac{4\bar{r}_0\chi\psi'(\bar{r}_0)}{1 + 2\bar{r}_0^2\chi} - \frac{\psi''(\bar{r}_0)}{2};$$

$$\alpha_2 = \frac{\sqrt{\psi'(\bar{r}_0)}}{\sqrt{1 + 2\bar{r}_0^2\chi}}\left(1 + \varepsilon\alpha_2^{(1)}\right) + O(\varepsilon^2);$$

$$\alpha_1 = \sqrt{\frac{F_0}{\bar{r}_0}}\left(1 - \varepsilon\alpha_1^{(1)}\right) + O(\varepsilon^2);$$

$$\alpha_1^{(1)} = \frac{1}{2}\left(S\bar{r}_0^2 + \frac{2D\bar{r}_0}{F_0} - \frac{D + F_0 S 3\bar{r}_0}{1 + 3\gamma_1\bar{r}_0^2 + 5g\bar{r}_0^4}\right);$$

$$\begin{aligned} \alpha_2^{(1)} = \frac{1}{2\psi'(\bar{r}_0)}\Bigg\{&3F_0 S\bar{r}_0 \\ &+\frac{(D + F_0 S 3\bar{r}_0)\left[1 + \bar{r}_0^2(6\chi - 3\gamma_1) + \bar{r}_0^4(6\gamma_1\chi - 15g) - 10g\bar{r}_0^6\chi\right]}{\left(1 + 3\gamma_1\bar{r}_0^2 + 5g\bar{r}_0^4\right)\left(1 + 2\bar{r}_0^2\chi\right)}\Bigg\}. \end{aligned}$$

The multiple scales method [32] is used to study the bifurcations of the system (36, 37). Then the solutions are represented by the asymptotic expansion:

$$\begin{aligned} &R = R_0(T_0, T_1, \ldots) + \varepsilon R_1(T_0, T_1, \ldots) + O(\varepsilon^2); \\ &\quad \phi = \phi_0(T_0, T_1, \ldots) + \varepsilon\phi_1(T_0, T_1, \ldots) + O(\varepsilon^2), \end{aligned} \tag{38}$$

where $T_0 = t; T_1 = \varepsilon t; T_2 = \varepsilon^2 t$. The following equations are derived:

$$\frac{\partial^2 R_0}{\partial T_0^2} + \alpha_2^2 R_0 = 0; \quad \frac{\partial^2\phi_0}{\partial T_0^2} + \alpha_1^2\phi_0 = 0;$$

$$\frac{\partial^2\phi_1}{\partial T_0^2} + \alpha_1^2\phi_1$$

$$= -2\frac{\partial^2\phi_0}{\partial T_0 \partial T_1} + \frac{F_0}{\bar{r}_0^2} R_0\phi_0 - \frac{2}{\bar{r}_0}\frac{\partial R_0}{\partial T_0}\frac{\partial \phi_0}{\partial T_0};$$

$$\frac{\partial^2 R_1}{\partial^2 T_0} + \alpha_2^2 R_1 = -2\frac{\partial^2 R_0}{\partial T_0 \partial T_1} + \frac{1}{1+2\chi\bar{r}_0^2} \left[G R_0^2 + C + \bar{r}_0\left(\frac{\partial \phi_0}{\partial T_0}\right)^2 - 2\chi\bar{r}_0\left(\frac{\partial R_0}{\partial T_0}\right)^2 - \frac{F_0}{2}\phi_0^2\right]. \quad (39)$$

The analysis of secular terms of the system (39) shows, that only one internal resonance exists:

$$\alpha_2 = 2\alpha_1 + \varepsilon\sigma, \quad (40)$$

where σ is the detuning parameter. Equating to zero the secular terms of the system (39), the following system of modulation equations is derived:

$$\alpha_2\frac{da_1}{dT_1} - \frac{B}{4}a_2^2\sin(\phi_1 + \sigma T_1 - 2\phi_2) = 0;$$
$$-\alpha_2 a_1\frac{d\phi_1}{dT_1} + \frac{B}{4}a_2^2\cos(\phi_1 - 2\phi_2 + \sigma T_1) = 0;$$
$$\alpha_1\frac{da_2}{dT_1} + \frac{\pi_*}{4}a_1 a_2\sin(\phi_1 - 2\phi_2 + \sigma T_1) = 0;$$
$$\alpha_1 a_2\frac{d\phi_2}{dT_1} + \frac{\pi_*}{4}a_1 a_2\cos(\phi_1 - 2\phi_2 + \sigma T_1) = 0, \quad (41)$$

where $B = \frac{1}{1+2\bar{r}_0^2\chi}(\alpha_1^2\bar{r}_0 + \frac{F_0}{2})$; $\pi_* = \frac{F_0}{\bar{r}_0^2} - \frac{2}{\bar{r}_0}\alpha_1\alpha_2$. Let us introduce the new variable $\psi = \phi_1 - 2\phi_2 + \sigma T_1$. Then the system (41) has the form:

$$\frac{da_1}{dT_1} = \frac{B}{4\alpha_2}a_2^2\sin\psi; \quad \frac{da_2}{dT_1} = \frac{\pi_*}{4\alpha_1}a_1 a_2\sin\psi;$$
$$\frac{d\psi}{dT_1} = \sigma + \left(\frac{B a_2^2}{4\alpha_2 a_1} + \frac{\pi_* a_1}{2\alpha_1}\right)\cos\psi. \quad (42)$$

The system (42) has two fixed points. The first one is determined by the equations:

$$a_1 = a_1^{(0)}; \qquad a_2 = 0; \qquad \cos\psi = -\frac{2\alpha_1\sigma}{\pi_2 a_1^{(0)}}, \quad (43)$$

where $a_1^{(0)}$ is an arbitrary constant. We stress, that formulae (43) describe the NNM (19, 28). The second fixed point is the following:

$$a_1 = a_1^{(0)}; \quad \sin\psi = 0; \quad \frac{B a_2^2}{4\alpha_2 a_1^{(0)}} = -\frac{\pi_* a_1^{(0)}}{2\alpha_1} \mp \sigma. \quad (44)$$

The fixed points are plotted on the frequency response $a_2(\sigma)$ (Fig. 4).

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the Jacobian matrix of the vector field (42) are determined to study stability of the fixed points. The fixed points (43) have the following values of $\lambda_1, \lambda_2, \lambda_3$:

$$\lambda_1 = 0; \quad \lambda_2 = -\frac{3\sqrt{F_0}a_1^{(0)}}{\bar{r}_0^{3/2}4}\sqrt{1 - \frac{4\sigma^2\alpha_1^2}{\pi_2^2 a_1^{(0)2}}};$$
$$\lambda_3 = -2\lambda_2. \quad (45)$$

If $1 - \frac{4\sigma^2\alpha_1^2}{\pi_2^2 a_1^{(0)2}} < 0$ $(1 - \frac{4\sigma^2\alpha_1^2}{\pi_2^2 a_1^{(0)2}} > 0)$, then the fixed point (43) is orbitally stable (unstable), respectively. Note that orbitally stable and unstable fixed points are shown on Fig. 4 by solid and dotted lines. The fixed point (44) has the following values of $\lambda_1, \lambda_2, \lambda_3$:

$$\lambda_1 = 0; \lambda_{2,3} = \pm i\frac{3a_2\sqrt{F_0\bar{r}_0}}{4\sqrt{1+2\bar{r}_0^2\chi}}\sqrt{\frac{1}{\bar{r}_0^2} + \frac{a_2^2}{16(1+2\bar{r}_0^2\chi)a_1^2}}. \quad (46)$$

Therefore, the fixed points (44) are orbitally stable. The frequency response (Fig. 4) contains two bifurcation points A, B, which have the following values of σ:

$$\sigma = \mp\frac{3\sqrt{F_0\bar{r}_0}a_1^{(0)}}{2\bar{r}_0^{3/2}}. \quad (47)$$

Let us study the bifurcation behavior on the plane (f_{10}, f_{20}). Such bifurcation set is described by the system of two nonlinear equations, which is obtained from (34, 47):

$$\bar{r}_0 + \gamma_1\bar{r}_0^3 + g\bar{r}_0^5 = F_0; \quad \alpha_2 = 2\alpha_1 \mp \varepsilon\frac{3\sqrt{F_0\bar{r}_0}a_1^{(0)}}{3\bar{r}_0^{3/2}}. \quad (48)$$

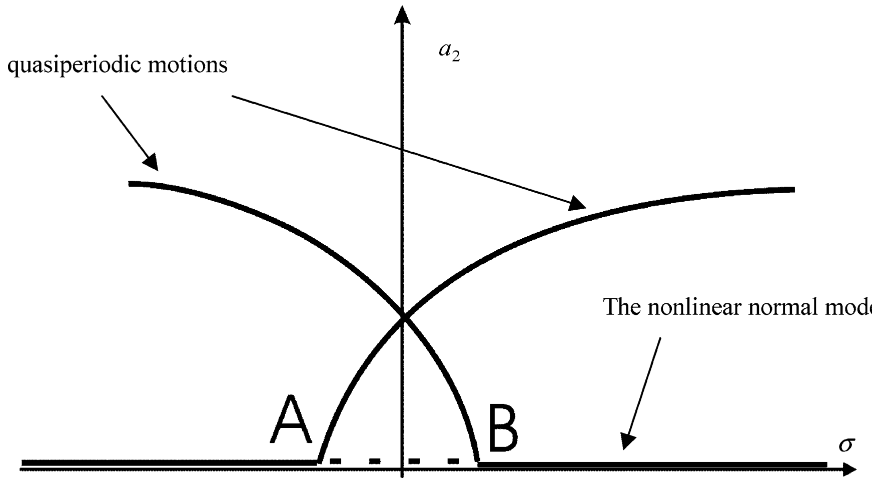


Fig. 4 The frequency response $a_2(\sigma)$. The stable motions are denoted by solid lines and unstable motions by dotted line

The following bifurcation set is derived by the asymptotic analysis:

$$S_B = \left\{(f_{10}, f_{20}) \in R^2 / \left[\psi(\bar{r}_0) + \varepsilon \bar{r}_1^{(\pm)} \psi'(\bar{r}_0)\right]^2 + O(\varepsilon^2) = f_{10}^2 + f_{20}^2\right\}, \tag{49}$$

where $\bar{r}_0$ is determined from the nonlinear equation:

$$3 + (\gamma_1 + 8\chi)\bar{r}_0^2 + (8\gamma_1\chi - g)\bar{r}_0^4 + 8\chi g \bar{r}_0^6 = 0.$$

Others parameters of Equation (49) are calculated in the following way:

$$\bar{r}_1^{(\pm)} = \frac{1}{\mathrm{P}_s(\bar{r}_0)}\left[\beta(\bar{r}_0) \pm \frac{6V(\bar{r}_0)a_1^{(0)}}{\bar{r}_0}\right],$$

$$\begin{aligned} V(r_0) &= 1 + \gamma_1 r_0^2 + g r_0^4 P_s(\bar{r}_0) \\ &= \left(1 + 2\chi\bar{r}_0^2\right)^{-2}\left[\bar{r}_0(2\gamma_1 + 4\chi) + \bar{r}_0^3(32\gamma_1\chi - 4g) + r_0^5(44g\chi + 32\chi^2\gamma_1) + 64g\chi^2 r_0^7\right]; \end{aligned}$$

$$P(\bar{r}_0) = -3 - \bar{r}_0^2(10\chi + 7\gamma_1) - \bar{r}_0^4(10\chi\gamma_1 + 19g + 16\chi^2) - \bar{r}_0^6(26g\chi + 16\chi^2\gamma_1) - 16g\chi^2\bar{r}_0^8;$$

$$\beta(\bar{r}_0) = \frac{3F_0 S\bar{r}_0}{1 + 2\chi\bar{r}_0^2} + 4\left(1 + \gamma_1 r_0^2 + g r_0^4\right) \times \left(S\bar{r}_0^2 + \frac{2D\bar{r}_0}{F_0}\right) + \frac{P(\bar{r}_0)(D + F_0 S 3\bar{r}_0)}{\left(1 + 3\gamma_1\bar{r}_0^2 + 5g\bar{r}_0^4\right)\left(1 + 2\bar{r}_0^2\chi\right)^2}.$$

Note, that the bifurcation set (49) has the form of two concentrically circles. The width of unstable area is estimated as $O(\varepsilon)$.

5. Analysis of internal resonance 1:2

The multiple scales method [32] is used to analyze the system (15) in the first approximation of μ. Solutions of the system (15) have the form:

$$x_j = x_{j0}(\tilde{T}_0, \tilde{T}_1, \ldots) + \mu x_{j1}(\tilde{T}_0, \tilde{T}_1, \ldots) + \cdots; \quad j = \overrightarrow{1, 3}, \tag{50}$$

where $\tilde{T}_0 = t; \tilde{T}_1 = \mu t; \ldots$ The expansion (50) is substituted into (15) and the terms of the same order ofμ are equated. As a result the following equations are derived:

$$x_{j0} = A_j(\tilde{T}_1)\exp(i\tilde{T}_0) + \bar{A}_j(\tilde{T}_1)\exp(-i\tilde{T}_0); \quad j = \overrightarrow{1, 3}; \tag{51}$$

$$\begin{aligned} &\frac{\partial^2 x_{11}}{\partial \tilde{T}_0^2} + x_{11} + \exp(i\tilde{T}_0)\left[2i\frac{\partial A_1}{\partial \tilde{T}_1} + \bar{\delta}_1 i A_1 + 3\bar{\gamma}_2 A_1^2 \bar{A}_1 + 2\bar{\gamma}_3 A_1 A_3 \bar{A}_3 + 2\bar{\gamma}_2 A_2 \bar{A}_2 A_1 + \bar{\gamma}_2 A_2^2 \bar{A}_1\right] + \bar{\gamma}_1 \bar{A}_1 A_3 \exp[i\tilde{T}_0(\bar{\omega}_3 - 1)] \\ &+ \bar{\gamma}_3 \bar{A}_1 A_3^2 \exp[i\tilde{T}_0(2\bar{\omega}_3 - 1)] = \frac{\bar{\alpha}_1}{2}\exp(ip\tilde{T}_0) + \cdots; \end{aligned} \tag{52}$$

$$\frac{\partial^2 x_{21}}{\partial \tilde{T}_0^2} + \exp(i\tilde{T}_0)\Big[2\bar{\gamma}_3 A_3 \bar{A}_3 A_2 + 2\bar{\gamma}_2 A_2 A_1 \bar{A}_1$$
$$+3\bar{\gamma}_2 A_2^2 \bar{A}_2 + \bar{\gamma}_2 \bar{A}_2 A_1^2 + 2i\frac{\partial A_2}{\partial \tilde{T}_1} + \bar{\delta}_2 i A_2\Big]$$
$$+x_{21} + \bar{\gamma}_3 \bar{A}_2 A_3^2 \exp(i\tilde{T}_0[2\bar{\omega}_3 - 1])$$
$$+\bar{\gamma}_1 \bar{A}_2 A_3 \exp[i\tilde{T}_0(\bar{\omega}_3 - 1)] + \cdots = 0; \qquad (53)$$

$$\frac{\partial^2 x_{31}}{\partial \tilde{T}_0^2} + \bar{\omega}_3^2 x_{31} + \exp(i\bar{\omega}_3\tilde{T}_0)$$
$$\left[2i\bar{\omega}_3\frac{\partial A_3}{\partial \tilde{T}_1} + \frac{4}{3}\bar{\gamma}_3(A_1\bar{A}_1 + A_2\bar{A}_2)A_3\right]$$
$$+\frac{\bar{\gamma}_1}{3}\left(A_1^2 + A_2^2\right)\exp(i2\tilde{T}_0)$$
$$+\frac{2}{3}\bar{\gamma}_3\left(A_1^2 + A_2^2\right)\bar{A}_3\exp[i\tilde{T}_0(2 - \bar{\omega}_3)] + \cdots = 0. \qquad (54)$$

Only essential terms are retained in the Equations (52–54). As follows from these equations five resonances can take place in the system (15):

$$a - p \approx 1; \quad b - \bar{\omega}_3 \approx 2; \quad c - \bar{\omega}_3 \approx 2; \quad p \approx 1;$$
$$d - \bar{\omega}_3 \approx 1; \quad e - \bar{\omega}_3 \approx 1; \quad p \approx 1.$$

The book [32] contains the approach of resonance determination for multiple scales method. The case $a(p \approx 1)$ corresponds to the main resonance in the system (15) and one internal resonance between eigenfrequencies of conjugate modes. The case $b(\bar{\omega}_3 \approx 2)$ corresponds to the internal resonance between eigenfrequency of axisymmetric mode $\bar{\omega}_3$ and eigenfrequency of conjugate modes, which is equal to 1 in dimensionless form. The case c merges the cases a and b. The case d corresponds to the forced oscillations of cylindrical shell with internal resonance between eigenfrequency of axisymmetric mode $\bar{\omega}_3$ and eigenfrequency of conjugate modes. It is suggested that the internal resonance between conjugate modes takes place. The case e merges the cases d and a. Kubenko, Koval'shuk and Krasnopolskaya [3] studied the case a. The cases b and c are considered in this paper.

The internal resonance 1:2 can be presented as

$$\bar{\omega}_3 = 2 + \varepsilon\sigma, \qquad (55)$$

where σ is a detuning parameter. Let us determine the shell parameters, which satisfy this internal resonance. Then only the summands of order O(1) are taken into account in (55) and the term $\varepsilon\sigma$ is rejected. The parameters of the system (15) are substituted into (55). Then the following fourth order algebraic equation with respect to ξ is obtained:

$$\frac{5n^2}{s^2}h^2\xi^4 - 16n^2h^2\xi^3 + \xi^2(8n^2h^2s^2 - 18)$$
$$+72s^2\xi - 36s^4 = 0, \qquad (56)$$

where $\xi = r^2 + s^2$. If the value ξ is calculated from (56), then the following formula for the shell length l is derived:

$$l = \frac{m\pi}{\sqrt{\xi - n^2R^{-2}}}. \qquad (57)$$

Now numerical values of the shell parameters, which satisfy the internal resonance (55), are determined. The values n, h, s are set and the real roots of Equation (56) are determined. Then the shell length l is calculated from (57). As a result, the following values of the shell parameters are calculated:

$$h = 8 \times 10^{-3}\,\text{m}; \quad \nu = 0.3; \quad R = 0.2\,\text{m};$$
$$\mu = 4 \times 10^{-2}; \quad n = 2; \quad l = 0.2\,\text{m}; \qquad (58)$$
$$E = 2 \times 10^{11}\,\text{Pa}; \quad \rho = 7.8 \times 10^3\,\text{kg/m}^3;$$
$$m = 1; \quad \bar{\omega}_3 = 2.013; \quad \bar{\gamma}_1 = -5.707;$$
$$\bar{\gamma}_2 = 0.68; \quad \gamma_3 = 0.619.$$

Now the nonlinear dynamics of shell in the case of internal resonance (55) is analyzed. The secular terms are excluded from (52–54). As a result the system of three modulation equations with respect to the complex variables is derived. The following change of the variables is used to the obtained system of modulation equations with respect to real variables: $A_j = 0.5a_j\exp\left(i\psi_j\right);\ j = \overrightarrow{1,3}$. As a result it is derived:

$$a_1' = -\frac{\bar{\delta}_1}{2}a_1 - \frac{\bar{\gamma}_2}{8}a_2^2a_1\sin(2\psi_2 - 2\psi_1)$$
$$-\frac{\bar{\gamma}_1}{4}a_1a_3\sin(\sigma\tilde{T}_1 + \psi_3 - 2\psi_1); \qquad (59)$$

$$\psi_1' = \frac{3\bar{\gamma}_2}{8}a_1^2 + \frac{\bar{\gamma}_3}{4}a_3^2 + \frac{\bar{\gamma}_2}{4}a_2^2 + \frac{\bar{\gamma}_2}{8}a_2^2\cos(2\psi_2 - 2\psi_1) + \frac{\bar{\gamma}_1}{4}a_3\cos(\sigma\tilde{T}_1 + \psi_3 - 2\psi_1); \quad (60)$$

$$a_2' = -\frac{\bar{\delta}_2}{2}a_2 - \frac{\bar{\gamma}_2}{8}a_1^2 a_2\sin(2\psi_1 - 2\psi_2) - \frac{\bar{\gamma}_1}{4}a_2 a_3\sin(\sigma\tilde{T}_1 + \psi_3 - 2\psi_2); \quad (61)$$

$$\psi_2' = \frac{\bar{\gamma}_3}{4}a_3^2 + \frac{\bar{\gamma}_2}{4}a_1^2 + \frac{3\bar{\gamma}_2}{8}a_2^2 + \frac{\bar{\gamma}_2}{8}a_1^2\cos(2\psi_1 - 2\psi_2) + \frac{\bar{\gamma}_1}{4}a_3\cos(\sigma\tilde{T}_1 + \psi_3 - 2\psi_2); \quad (62)$$

$$a_3' = -\frac{\bar{\delta}_3}{2}a_3 - \frac{\bar{\gamma}_1}{12\bar{\omega}_3}a_1^2\sin(2\psi_1 - \sigma\tilde{T}_1 - \psi_3) - \frac{\bar{\gamma}_1 a_2^2}{12\bar{\omega}_3}\sin(2\psi_2 - \psi_3 - \sigma\tilde{T}_1); \quad (63)$$

$$\psi_3' = \frac{\bar{\gamma}_3}{6\bar{\omega}_3}\left(a_1^2 + a_2^2\right) + \frac{\bar{\gamma}_1 a_1^2}{12\bar{\omega}_3 a_3}\cos(2\psi_1 - \sigma\tilde{T}_1 - \psi_3) + \frac{\bar{\gamma}_1 a_2^2}{12\bar{\omega}_3 a_3}\cos(2\psi_2 - \psi_3 - \sigma\tilde{T}_1). \quad (64)$$

where $(\cdot)' = \frac{d(\cdot)}{d\tilde{T}_1}$. All parameters of the system (59–64) are defined in Section 2 and Appendix. The system (59–64) is reduced to the five equations by the change of the variables $(a_1, a_2, a_3, \theta_1, \theta_2) = (a_1, a_2, a_3, 2\psi_1 - \sigma\tilde{T}_1 - \psi_3, 2\psi_2 - \psi_3 - \sigma\tilde{T}_1)$:

$$\begin{aligned}
a_1' &= -\frac{\bar{\delta}_1}{2}a_1 - \frac{\bar{\gamma}_2}{8}a_2^2 a_1\sin(\theta_2 - \theta_1) - \frac{\bar{\gamma}_1}{4}a_1 a_3\sin\theta_1;\\
a_2' &= -\frac{\bar{\delta}_2}{2}a_2 - \frac{\bar{\gamma}_2}{8}a_1^2 a_2\sin(\theta_2 - \theta_1) + \frac{\bar{\gamma}_1}{4}a_2 a_3\sin\theta_2;\\
a_3' &= -\frac{\bar{\delta}_3}{2}a_3 - \frac{\bar{\gamma}_1}{24}a_1^2\sin\theta_1 - \frac{\bar{\gamma}_1 a_2^2}{24}\sin\theta_2;\\
\theta_1' &= -\sigma + \frac{3\bar{\gamma}_2}{4}a_1^2 + \frac{\bar{\gamma}_3}{2}a_3^2 + \frac{\bar{\gamma}_2}{2}a_2^2 + \frac{\bar{\gamma}_2}{4}a_2^2\cos(\theta_2 - \theta_1) + \frac{\bar{\gamma}_1}{2}a_3\cos\theta_1\\
&\quad - \frac{\bar{\gamma}_1 a_1^2}{24 a_3}\cos\theta_1 - \frac{\bar{\gamma}_1 a_2^2}{24 a_3}\cos\theta_2;\\
\theta_2' &= \frac{\bar{\gamma}_3}{2}a_3^2 + \frac{\bar{\gamma}_2}{2}a_1^2 + \frac{3\bar{\gamma}_2}{4}a_2^2 + \frac{\bar{\gamma}_2}{4}a_1^2\cos(\theta_2 - \theta_1)\\
&\quad + \frac{\bar{\gamma}_1}{2}a_3\cos\theta_2 - \frac{\bar{\gamma}_1 a_1^2}{24 a_3}\cos\theta_1 - \frac{\bar{\gamma}_1 a_2^2}{24 a_3}\cos\theta_2 - \sigma.
\end{aligned} \quad (65)$$

Only one fixed point $a_1 = a_2 = a_3 = 0$ exists in the dynamical system (65). The linearized flow of the vector field (65) in the point $a_1 = a_2 = a_3 = 0$ is derived to study stability. Eigenvalues of this linearized flow are

$$\lambda_1 = -\frac{\bar{\delta}_1}{2}; \quad \lambda_2 = -\frac{\bar{\delta}_2}{2}; \quad \lambda_{3,4} = -\frac{\bar{\delta}_3}{2}; \quad \lambda_5 = 0. \quad (66)$$

Hence, the fixed point $a_1 = a_2 = a_3 = 0$ is stable. These results have the following physical meaning. If the resonance for p is not taken place and the internal resonance (55) is considered, then the general coordinates of the system (15) x_1, x_2, x_3 have orders $O(\mu^2)$.

6. Primary resonance of forced oscillations

In this section the case with the internal resonance (55) and the primary resonance

$$p = 1 + \mu\varsigma \quad (67)$$

is considered. Annihilating the secular terms in (52, 53, 54), the system of modulation equations with respect to $(a_1, a_2, a_3, \theta_1, \theta_2, \theta_3) = (a_1, a_2, a_3, \varsigma\tilde{T}_1 - \psi_1; 2\psi_2 - 2\varsigma\tilde{T}_1; \psi_3 + \sigma\tilde{T}_1 - 2\varsigma\tilde{T}_1)$ is derived:

$$\begin{aligned}
a_1' &= -\frac{\bar{\delta}_1}{2}a_1 - \frac{\bar{\gamma}_2}{8}a_2^2 a_1\sin(\theta_2 + 2\theta_1) - \frac{\bar{\gamma}_1}{4}a_1 a_3\sin(\theta_3 + 2\theta_1) + \frac{\bar{\alpha}_1}{2}\sin(\theta_1);\\
a_1\theta_1' &= \varsigma a_1 - \frac{3\bar{\gamma}_2}{8}a_1^3 - \frac{\bar{\gamma}_3}{4}a_1 a_3^2 - \frac{\bar{\gamma}_2}{4}a_2^2 a_1 - \frac{\bar{\gamma}_2}{8}a_2^2 a_1\cos(\theta_2 + 2\theta_1)\\
&\quad - \frac{\bar{\gamma}_1}{4}a_1 a_3\cos(\theta_3 + 2\theta_1) + \frac{\bar{\alpha}_1}{2}\cos(\theta_1);\\
a_2' &= -\frac{\bar{\delta}_2}{2}a_2 + \frac{\bar{\gamma}_2}{8}a_1^2 a_2\sin(\theta_2 + 2\theta_1) - \frac{\bar{\gamma}_1}{4}a_2 a_3\sin(\theta_3 - \theta_2);\\
\theta_2' a_2 &= -2\varsigma a_2 + \frac{\bar{\gamma}_3}{2}a_3^2 a_2 + \frac{\bar{\gamma}_2}{2}a_1^2 a_2 + \frac{3\bar{\gamma}_2}{4}a_2^3 + \frac{\bar{\gamma}_2}{4}a_1^2 a_2\cos(\theta_2 + 2\theta_1)
\end{aligned}$$

$$+\frac{\bar{\gamma}_1}{2}a_3a_2\cos(\theta_3-\theta_2);$$

$$a_3' = -\frac{\bar{\delta}_3}{2}a_3 + \frac{\bar{\gamma}_1}{24}a_1^2\sin(2\theta_1+\theta_3)$$

$$+\frac{\bar{\gamma}_1 a_2^2}{24}\sin(\theta_3-\theta_2);$$

$$a_3\theta_3' = (\sigma-2\varsigma)a_3 + \frac{\bar{\gamma}_3}{12}\left(a_1^2+a_2^2\right)a_3$$

$$+\frac{\bar{\gamma}_1 a_1^2}{24}\cos(\theta_3+2\theta_1) + \frac{\bar{\gamma}_1 a_2^2}{24}\cos(\theta_2-\theta_3), \tag{68}$$

where ψ_1, ψ_2, ψ_3 are the variables of dynamical system (59–64).

6.1. Undamped case

Let us consider the cylindrical shell oscillations without damping: $\bar{\delta}_1 = \bar{\delta}_2 = \bar{\delta}_3 = 0$. Now the standing waves, which correspond to the fixed points of the system (68) ($a_2 = 0$; $\theta_2(t) \neq 0$), are analyzed,. These fixed points are described by the equations:

$$-\varsigma + \frac{3}{8}\bar{\gamma}_2 a_1^2 + \frac{\bar{\gamma}_3}{4}a_3^2 + \frac{\bar{\gamma}_1}{4}a_3(-1)^n - \frac{\bar{\alpha}_1}{2a_1}(-1)^m = 0;$$

$$\sigma - 2\varsigma + \frac{\bar{\gamma}_3}{12}a_1^2 + \frac{\bar{\gamma}_1}{24a_3}a_1^2(-1)^n = 0, \tag{69}$$

where m and n are integers.

The frequency response $a_1(\varsigma)$, $a_2(\varsigma)$ of the standing waves is determined. The case $\bar{\alpha}_1 \ll 1$ is considered. Then the zero approximation of the system (69) ($\bar{\alpha}_1 = 0$) are described by the following equations:

$$\varsigma = \frac{9\bar{\gamma}_2\sigma - 0.5\bar{\gamma}_3^2a_3^2 - 0.75\bar{\gamma}_3\bar{\gamma}_1(-1)^n a_3 - 0.25\bar{\gamma}_1^2}{18\bar{\gamma}_2 - 2\bar{\gamma}_3 - \bar{\gamma}_1(-1)^n a_3^{-1}};$$

$$a_1 = \sqrt{\frac{24a_3(2\varsigma-\sigma)}{2\bar{\gamma}_3 a_3 + \bar{\gamma}_1(-1)^n}}. \tag{70}$$

The calculations with the parameters (58) and $\sigma = 2.5$ are carried out according to the Equation (70). The results are presented on Fig. 5, where a_1 is shown versus ς.

Let us consider the case of an arbitrary value of $\bar{\alpha}_1$. Then the following cubic equation is derived from (69):

$$-144z^3 + P_1z^2 + P_2z + P_3 = 0; \tag{71}$$

$$P_1 = 24\bar{\gamma}_3a_1^2 + 144\left[\frac{3}{4}\bar{\gamma}_2a_1^2 - \sigma - \frac{\bar{\alpha}_1}{a_1}(-1)^m\right];$$

$$P_2 = 3\bar{\gamma}_1^2a_1^2 - \bar{\gamma}_3^2a_1^4 - 24\bar{\gamma}_3a_1^2\left[\frac{3}{4}\bar{\gamma}_2a_1^2 - \sigma - \frac{\bar{\alpha}_1}{a_1}(-1)^m\right];$$

$$P_3 = \bar{\gamma}_3^2a_1^4\left[\frac{3}{4}\bar{\gamma}_2a_1^2 - \sigma - \frac{\bar{\alpha}_1}{a_1}(-1)^m\right] - \frac{\bar{\gamma}_3}{8}\bar{\gamma}_1^2a_1^4.$$

The frequency response is calculated according to the Equation (71) with the parameters (58). Given the value

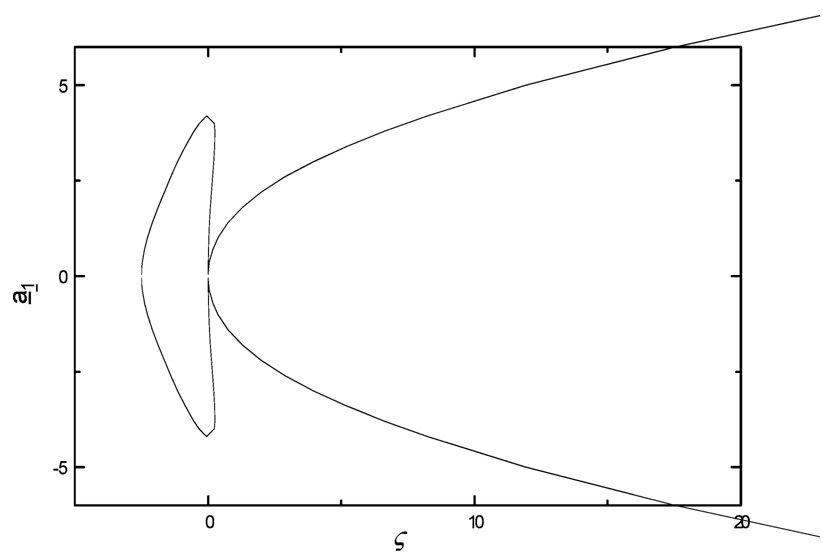


Fig. 5 Zero approximation of the frequency response of system (15) at $\bar{\alpha}_1 = 0$. The value a_1 is shown versus z

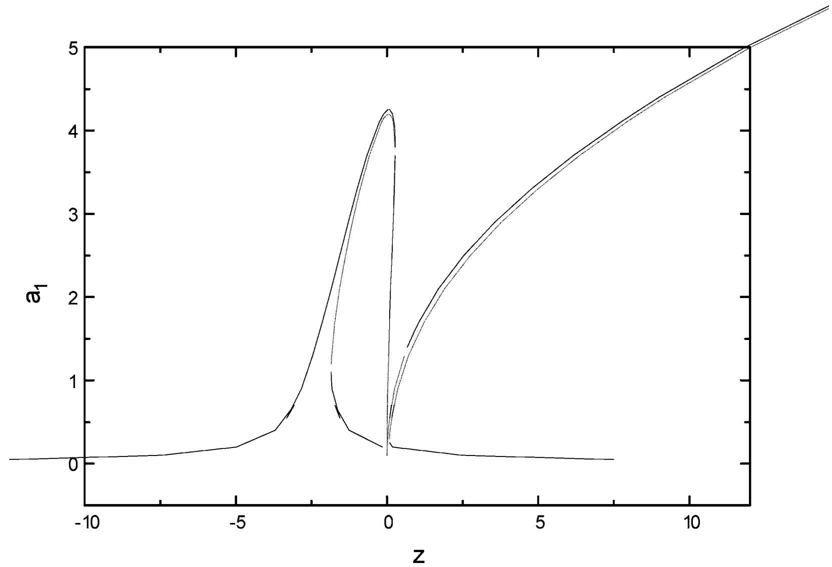


Fig. 6 The frequency response of standing waves in undamped shell. Stable (unstable) oscillations are shown by solid (dotted) lines, respectively

a_1 with the certain step, the cubic equation is solved. The frequency response is shown on Fig. 6.

Now the function $\theta_2(t)$ (see Equation (68)) is determined to analyze a stability of standing waves. The function $\theta_2(t)$ satisfies the following equation:

$$\theta_2' = C_1 + C_2 \cos\theta_2, \tag{72}$$

where $C_1 = -2\varsigma + 0.5\bar{\gamma}_3 a_3^2 + 0.5\bar{\gamma}_2 a_1^2$; $C_2 = 0.25\bar{\gamma}_2 a_1^2 + 0.5\bar{\gamma}_1 a_3(-1)^n$. If $-1 < C_1 C_2^{-1} < 1$, the Equation (72) has the fixed point. The analytical solution of (72) is the following:

$$\theta_2(t) = \begin{cases} 2\arctan\left\{\sqrt{\frac{C_1+C_2}{C_1-C_2}}\tan\left[\frac{1}{2}\sqrt{C_1^2 - C_2^2}(t-t_0)\right]\right\}; & |C_1| > |C_2|; \\ 2\arctan\left\{\sqrt{\frac{C_1+C_2}{C_2-C_1}}\tanh\left[\frac{1}{2}\sqrt{C_2^2 - C_1^2}(t-t_0)\right]\right\}; & |C_1| < |C_2|. \end{cases} \tag{73}$$

Note, that the function $\theta_2(t)$ is growing, if $|C_1| > |C_2|$ and it tends to fixed point, if $|C_1| < |C_2|$.

The small deviations $\Delta a_1, \Delta\theta_1, \Delta a_2, \Delta\theta_2, \Delta a_3, \Delta\theta_3$ from the fixed points are considered to study stability. The system of variation equations has the following form:

$$\begin{bmatrix} \Delta a_1' \\ \Delta\theta_1' \\ \Delta a_3' \\ \Delta\theta_3' \end{bmatrix} = \begin{bmatrix} 0 & B_{12} & 0 & B_{14} \\ B_{21} & 0 & B_{23} & 0 \\ 0 & B_{32} & 0 & B_{34} \\ B_{41} & 0 & B_{43} & 0 \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta\theta_1 \\ \Delta a_3 \\ \Delta\theta_3 \end{bmatrix}; \tag{74}$$

$$\Delta a_2' = \chi \sin\theta_2 \Delta a_2; \tag{75}$$

$$\begin{aligned} \Delta\theta_2' = {} & A_1(t)\Delta\theta_2 + A_2(t)\Delta a_1 + A_3(t)\Delta a_3 \\ & + A_4(t)\Delta\theta_1 + A_5(t)\Delta\theta_3, \end{aligned} \tag{76}$$

where the coefficients $B_{12}, \ldots, B_{43}$, $\chi, A_1(t), \ldots, A_5(t)$ are given in Appendix. Note that the system (74) is independent on the Equations (75) and (76). The Equation (76) is coupled with the system (74) and the Equation (76) can be solved after (74). The characteristic exponents $\lambda_1, \ldots, \lambda_4$ of the system (74) are determined as

$$\begin{vmatrix} -\lambda & B_{12} & 0 & B_{14} \\ B_{21} & -\lambda & B_{23} & 0 \\ 0 & B_{32} & -\lambda & B_{34} \\ B_{41} & 0 & B_{43} & -\lambda \end{vmatrix} = 0. \tag{77}$$

The Equation (77) can be presented as

$$\lambda^4 - \lambda^2 b + c = 0, \tag{78}$$

$$\begin{aligned} c = {} & \frac{\bar{\gamma}_1\bar{\alpha}_1}{48} a_1^2 (-1)^{n+m} \left[\frac{\bar{\gamma}_3^2 a_1 a_3}{12} + \frac{\bar{\gamma}_1^2 a_1}{48 a_3} + \frac{\bar{\gamma}_3\bar{\gamma}_1 a_1(-1)^n}{12} \right. \\ & \left. + \frac{\bar{\gamma}_1\bar{\gamma}_2 a_1^3(-1)^n}{32 a_3^2} + \frac{\bar{\gamma}_1\bar{\alpha}_1(-1)^{n+m}}{48 a_3^2}\right]; \end{aligned}$$

$$b = \frac{3}{8}\bar{\gamma}_2\bar{\gamma}_1 a_1^2 a_3(-1)^n - \frac{3\bar{\alpha}_1}{8}(-1)^m\bar{\gamma}_2 a_1 + \frac{\bar{\alpha}_1\bar{\gamma}_1 a_3}{4a_1}(-1)^{m+n} - \frac{\bar{\alpha}_1^2}{4a_1^2} - \frac{\bar{\gamma}_1\bar{\gamma}_3}{12}a_1^2 a_3(-1)^n - \frac{\bar{\gamma}_1^2 a_1^4}{576 a_3^2} - \frac{\bar{\gamma}_1^2 a_1^2}{24}. \quad (83)$$

The solution of the Equation (75) is the following:

$$\Delta a_2 = \Delta a_{20}\exp\left(\chi\int_0^t \sin\theta_2 dt\right), \quad (79)$$

where $\Delta a_2(0) = \Delta a_{20}$ is the initial condition; the constant χ is determined in Appendix. The integral (79) is calculated using the Equation (72). The solution of Equation (79) can be presented as

$$\Delta a_2^2(t) = \Delta a_{20}^2\left|\frac{C_1 + C_2\cos\theta_2^{(0)}}{C_1 + C_2\cos\theta_2(t)}\right|, \quad (80)$$

where $\theta_2^{(0)}$ is an initial condition for the Equation (72).

If solutions of the system (72) tend to the fixed point and the denominator of (80) tends to zero, then the variable Δa_2 increases infinitely. In this case the standing waves are unstable. Thus, as follow from the expression (80), the condition of instability of the standing waves is $\left|C_1 C_2^{-1}\right| < 1$. This inequality can be presented as

$$\left|\frac{-2\varsigma + 0.5\bar{\gamma}_3 a_3^2 + 0.5\bar{\gamma}_2 a_1^2}{0.25\bar{\gamma}_2 a_1^2 + 0.5\bar{\gamma}_1 a_3(-1)^n}\right| < 1. \quad (81)$$

The following function is introduced to study the Equation (76):

$$F_\Sigma(t) = \alpha_1\Delta a_1 + \alpha_2\Delta a_3 + (\alpha_3\Delta a_1 + \alpha_4\Delta a_3)\cos\theta_2 + (\alpha_5\Delta\theta_1 + \alpha_6\Delta\theta_3)\sin\theta_2 = \sum \bar{P}_i(t)\exp(\lambda_i t);$$
$$\bar{P}_i(t) = A_i^{(\Sigma)} + B_i^{(\Sigma)}\cos\theta_2 + D_i^{(\Sigma)}\sin\theta_2, \quad (82)$$

where the values $\alpha_1, \ldots, \alpha_6$ are presented in Appendix. The parameters $A_i^{(\Sigma)}, B_i^{(\Sigma)}, D_i^{(\Sigma)}$ are linear functions of initial conditions of the system (74):

$$\begin{bmatrix} A_i^{(\Sigma)} \\ B_i^{(\Sigma)} \\ D_i^{(\Sigma)} \end{bmatrix} = \begin{bmatrix} \beta_{i,1}^{(1)} & \beta_{i,2}^{(1)} & \beta_{i,3}^{(1)} & \beta_{i,4}^{(1)} \\ \beta_{i,1}^{(2)} & \beta_{i,2}^{(2)} & \beta_{i,3}^{(2)} & \beta_{i,4}^{(2)} \\ \beta_{i,1}^{(3)} & \beta_{i,2}^{(3)} & \beta_{i,3}^{(3)} & \beta_{i,4}^{(3)} \end{bmatrix} \begin{bmatrix} \Delta a_1^{(0)} \\ \Delta\theta_1^{(0)} \\ \Delta a_3^{(0)} \\ \Delta\theta_3^{(0)} \end{bmatrix},$$

Solution of the Equation (76) is:

$$\Delta\theta_2 = \Delta\theta_2^{(0)}\exp\left(-\chi_2\int_0^t \sin\theta_2 d\tau\right) + \sum_{i=1}^{4}\int_0^t \bar{P}_i(\tau)\exp(\lambda_i\tau)\exp\left(-\chi_2\int_\tau^t \sin\theta_2 dt_1\right)d\tau, \quad (84)$$

where $\Delta\theta_2(0) = \Delta\theta_2^{(0)}$ is an initial condition. The Equation (72) permits to determine the integrals in the formula (84). Then the Equation (84) can be presented as

$$\Delta\theta_2 = \Delta\theta_2^{(0)}\left|\frac{C_1 + C_2\cos\theta_2(t)}{C_1 + C_2\cos\theta_2^{(0)}}\right| + R_I(t);$$
$$R_I(t) = \int_0^t F_\Sigma(\tau)\left|\frac{C_1 + C_2\cos\theta_2(t)}{C_1 + C_2\cos\theta_2(\tau)}\right| d\tau. \quad (85)$$

It is assumed that all characteristic exponents (78) have negative real parts and there is not fixed point of the Equation (73). Then the following estimation is true for the integral (85):

$$|R_I(t)| = |C_1 + C_2\cos\theta_2(t)|\left|\int_0^t \frac{\sum_i \bar{P}_i(t)\exp(\lambda_i t)}{|C_1 + C_2\cos\theta_2(\xi)|}d\xi\right| \leq \bar{\chi}^{(\Sigma)}\sum_i\int_0^t \exp(\mathrm{Re}[\lambda_i]\xi)d\xi. \quad (86)$$

Note, that the value $R_I(t)$ is bounded.

On the basis of the above-presented analysis the conditions of stability/instability of standing waves can be formulated in the following form. If the inequality (81) is not fulfilled, and all characteristic exponents $\lambda_1, \ldots, \lambda_4$ have negative real parts, the variable $\Delta\theta_2$ is limited and the standing waves are stable. If the condition of instability (81) is satisfied or one of the characteristic exponents $\lambda_1, \ldots, \lambda_4$ has positive real parts, the standing waves are unstable.

The stable and unstable solutions are shown on Fig. 6 by solid and dotted lines, respectively. Note, that this frequency response (Fig. 6) describes the standing wave in a cylindrical shell. The soft and hard frequency responses of cylindrical shells are considered in the paper [22]. Bondarenko, Telalov [33] showed experimentally, that the frequency response is hard for small number of waves in the circumference direction.

Now the traveling waves in cylindrical shells are considered. These waves are described by the fixed points of the modulation Equations (68), which satisfy the following relation: $a_i' = \theta_i' = 0; i = 1, 2, 3; a_1 \neq 0; a_2 \neq 0; a_3 \neq 0$. These fixed points satisfy the system of six nonlinear algebraic equations, which follows from (68):

$$Y(a_1, a_2, a_3, \theta_1, \theta_2, \theta_3, \varsigma) = 0. \tag{87}$$

The frequency response $a_1(\varsigma), a_2(\varsigma), a_3(\varsigma)$ is the solution of the system (87). To obtain the frequency response, the system (87) is studied by the continuation algorithm [34]. The calculations are performed with the parameters (58) and $\sigma = 0$. Fig. 7 shows the result of calculation $a_1(\varsigma)$.

The stability analysis is performed for fixed points (Fig. 7). The characteristic exponents $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_6$ are calculated from the equation:

$$\det[DY - \tilde{\lambda}E] = 0,$$

where DY is Jacobian matrix evaluated at the fixed points of the system (87); E is the identity matrix. The stable and unstable fixed points are shown on Fig. 7 by solid and dotted lines, respectively. Two saddle- node bifurcation points and two Andronov- Hopf bifurcation points are presented on Fig. 7.

6.2. Damped case

Nonlinear oscillations of damped shell are described by the system of modulation Equation (68). It is known from the experimental data [1, 3], that two conjugate modes make the main contribution into cylindrical shell nonlinear oscillations. Therefore, the work of a friction force of the conjugate modes is greater than the one of axisymmetric mode. Therefore, the damping coefficients are taken as $\bar{\delta}_1 = \bar{\delta}_2 = \bar{\delta}_*; \bar{\delta}_3 = 0$.

The standing waves in the shell are considered, which correspond to the fixed points $(a_2 = 0; \theta_2(t) \neq 0)$ of the system (68). In this case the frequency response is described by the cubic equation, which is similar to the Equation (71):

$$-144z^3 + Q_1 z^2 + Q_2 z + Q_3 = 0, \tag{88}$$

where $Q_1 = 24\bar{\gamma}_3 a_1^2 + 144 g_1$; $Q_2 = 3\bar{\gamma}_1^2 a_1^2 - 24 a_1^2 g_1 \bar{\gamma}_3 - \bar{\gamma}_3^2 a_1^4$; $Q_3 = g_1 \bar{\gamma}_3^2 a_1^4 - \frac{\bar{\gamma}_3 \bar{\gamma}_1^2 a_1^4}{8}$; $g_1 = -\sigma + \frac{3}{4}\bar{\gamma}_2 a_1^2 - (-1)^m \frac{\bar{\alpha}_1}{a_1}\sqrt{1 - \frac{\bar{\delta}_*^2}{\bar{\alpha}_1^2} a_1^2}$.The frequency response is calculated with the parameters (58) and $\bar{\delta}_* = 0.1$. Fig. 8 shows the frequency response $a_1(z)$.

The variation equations with respect to $(\Delta a_1, \Delta\theta_1, \Delta a_2, \Delta\theta_2, \Delta a_3, \Delta\theta_3)$ are derived to study a stability of standing wave. The system of variation equations in the damped case differs from

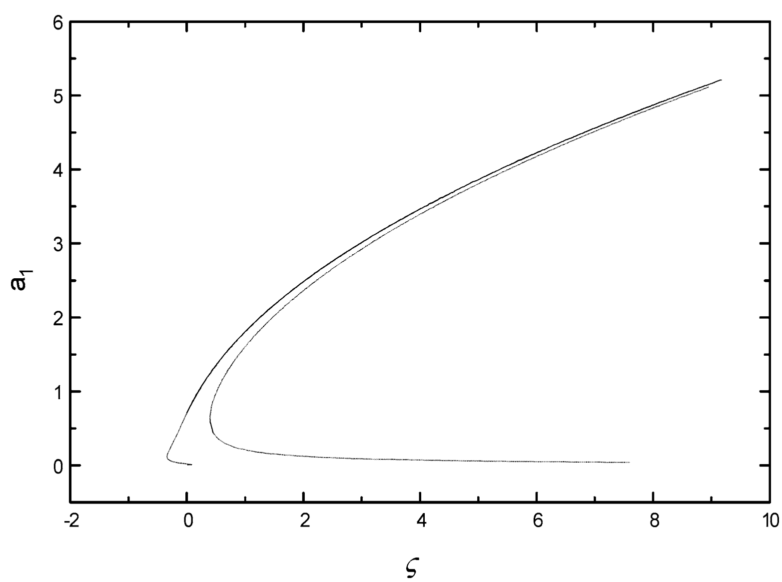


Fig. 7 The frequency response of traveling waves in undamped shell. Stable (unstable) oscillations are shown by solid (dotted) lines, respectively

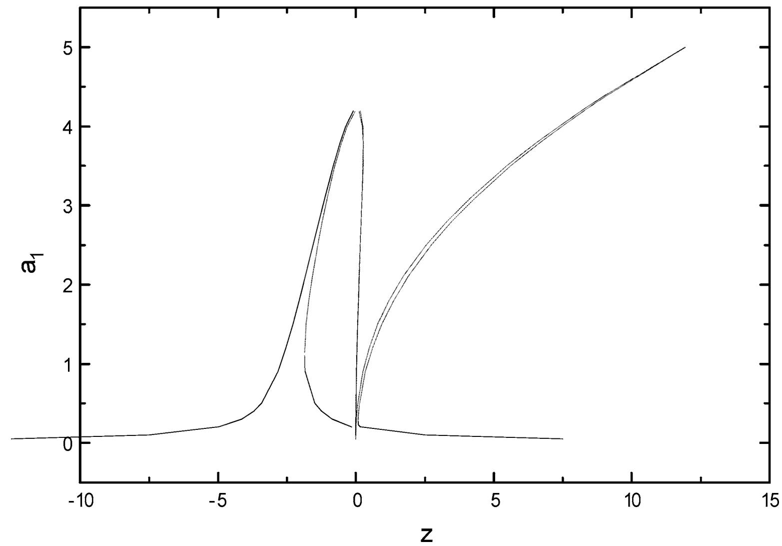


Fig. 8 The frequency response of standing waves in damped shell. Stable (unstable) oscillations are shown by solid (dotted) lines, respectively

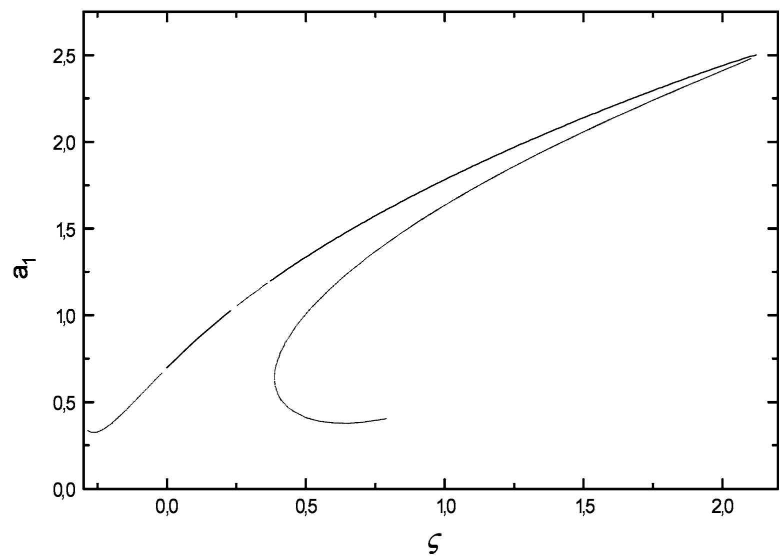


Fig. 9 The frequency response of traveling waves in damped shell. Stable (unstable) oscillations are shown by solid (dotted) lines, respectively

the Equations (74–76) only by dissipative terms and it has the form:

$$\begin{bmatrix} \Delta a_1' \\ \Delta\theta_1' \\ \Delta a_3' \\ \Delta\theta_3' \end{bmatrix} = \begin{bmatrix} -0.5\bar{\delta}_* & B_{12} & 0 & B_{14} \\ B_{21} & 0 & B_{23} & 0 \\ 0 & B_{32} & 0 & B_{34} \\ B_{41} & 0 & B_{43} & 0 \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta\theta_1 \\ \Delta a_3 \\ \Delta\theta_3 \end{bmatrix}; \tag{89}$$

$$\Delta a_2' = (-0.5\bar{\delta}_* + \chi \sin\theta_2)\Delta a_2; \tag{90}$$

$$\Delta\theta_2' = A_1(t)\Delta\theta_2 + A_2(t)\Delta a_1 + A_3(t)\Delta a_3$$
$$+A_4(t)\Delta\theta_1 + A_5(t)\Delta\theta_3. \tag{91}$$

The change of the variables is introduced to study the Equation (90)

$$\Delta a_2 = \exp\left(-0.5\bar{\delta}_\bullet t\right)\eta(t). \tag{92}$$

Then the Equation (90) takes the form:

$$\eta' = \chi \sin\theta_2 \eta. \tag{93}$$

The Equation (93) is the same as the Equation (75). Solutions of the Equation (90) can be presented as

$$\Delta a_2 = \Delta a_{20} \exp\left(-0.5\bar{\delta}_{\bullet} t + \chi \int_0^t \sin\theta_2 \, dt\right). \quad (94)$$

The equation for the variable θ_2 in the damped case coincides with the Equation (72). Repeating the transformation to calculate the integral in (94) (see Subsection 6.1), the solution of (90) can be presented as

$$\Delta a_2 = \Delta a_{20} \exp\left(-\frac{\bar{\delta}_*}{2} t\right) \left|\frac{C_1 + C_2 \cos\theta_2^{(0)}}{C_1 + C_2 \cos\theta(t)}\right|. \quad (95)$$

Thus, if the function

$$r(t) = \exp\left[-\frac{\bar{\delta}_*}{2} t\right] \cdot \left|\frac{C_1 + C_2 \cos\theta_2^{(0)}}{C_1 + C_2 \cos\theta_2(t)}\right|; \quad (96)$$

is bounded, the trivial solution of the Equation (90) is stable. As follows from the relation (96) and condition (80), if

$$\left|C_1 C_2^{-1}\right| > 1, \quad (97)$$

the trivial solution of the Equation (90) is asymptotically stable. If the Equation (72) has not fixed points, the trivial solution of the Equation (90) is asymptotically stable too.

Note that the Equation (91) coincides with the Equation (76) for the undamped case. Therefore, the conclusions about stability with respect to the variable $\theta_2(t)$ for the damped case are the same as for the undamped one.

To analyze a stability of the trivial solution of the system (89) the characteristic exponents $\lambda_1, \ldots, \lambda_4$ are determined from the following equation:

$$\lambda^4 + \frac{\bar{\delta}_*}{2}\lambda^3 - b\lambda^2 - \frac{\bar{\delta}_*}{2}\tilde{a}\lambda + c = 0, \quad (98)$$

where $\tilde{a} = -\frac{\bar{\gamma}_1^2 a_1^4}{576 a_3^2} - \frac{\bar{\gamma}_1 \bar{\gamma}_3 a_1^2 a_3}{24}(-1)^n - \frac{\bar{\gamma}_1^2 a_1^2}{48}$. The results of a stability analysis are presented on Fig. 8 for the system parameters (58) and $\bar{\delta}_* = 0.1$. Stable and unstable oscillations are shown by solid and dotted line, respectively.

Now the traveling waves in damped cylindrical shell are considered. The fixed points ($a_1 \neq 0; a_2 \neq 0; a_3 \neq 0$) of the modulation Equations (68) are described the traveling waves. The approach for these fixed points analysis in the undamped case is considered above. The method for the damped system investigations is the same as for the damped one.

The frequency response $a_1(\varsigma)$ of the damped system is presented on Fig. 9 for the shell parameters (58) and $\bar{\delta}_* = 0.1$, $\sigma = 0$. The fixed points (Fig. 9) are abounding in bifurcations. The Andronov- Hopf bifurcation

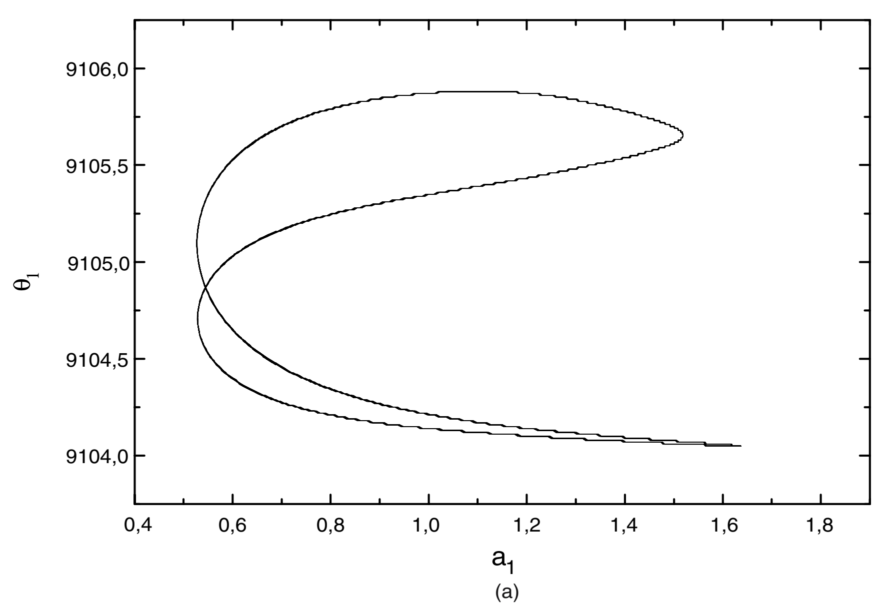
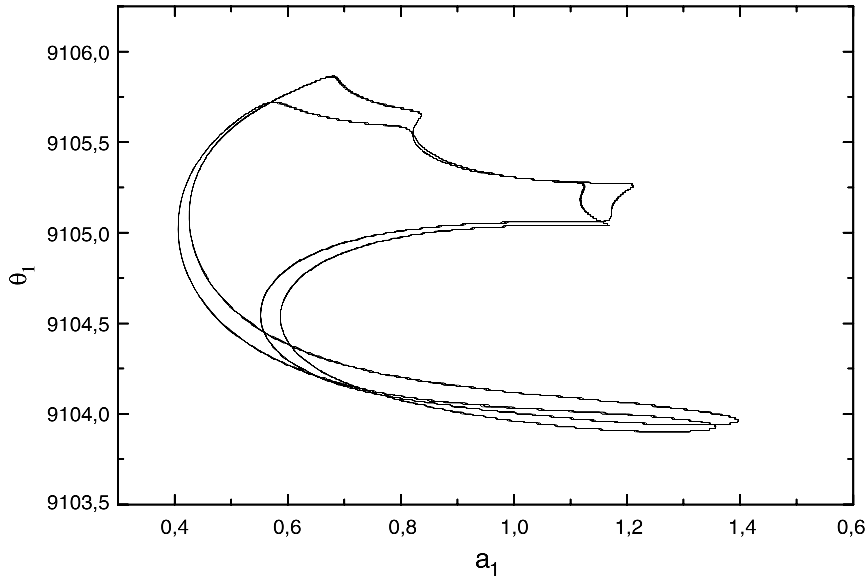
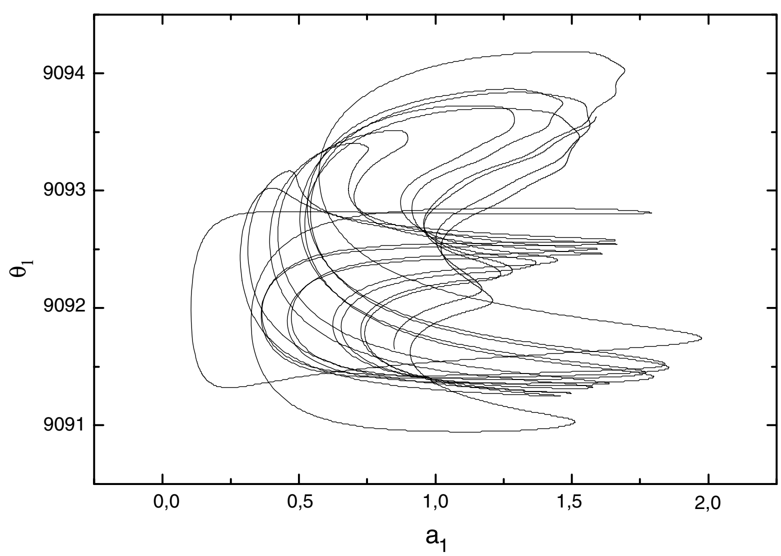


Fig. 10 The steady states in the plane (a_1, θ_1). These motions correspond to the following values of the detuning parameter: a- $\varsigma = -0.1465$; b- $\varsigma = -0.195$; c- $\varsigma = -0.22$ (*Continued on next page*)

Fig. 10 (*Continued*)



(b)



(c)

of the fixed points takes place at $\varsigma = 6.0 \cdot 10^{-4}$. As a result of this bifurcation, the limit cycle appears. Such limit cycle is shown on Fig. 10a at $\varsigma = -0.1465$. This cycle undergoes the periodic doubling bifurcation. Figure 10b shows the periodic motions on the plane (a_1, θ_1) after this bifurcation at $\varsigma = -0.195$. After the sequence of periodic doubling bifurcation chaotic motions take place. The chaotic motions at $\varsigma = -0.22$ are shown on Fig. 10c.

7. Conclusions

The results of the nonlinear normal mode analysis for cylindrical shells are presented in this paper. Only a single nonlinear normal vibration mode close to a straight line is discovered. This NNM is mainly determined by the shell imperfections. This mode undergoes the bifurcations and the stable quasiperiodic motions appear.

The forced oscillations of circular cylindrical shell in the case of two internal resonances are analyzed in this paper. One condition of the internal resonance describes equality of the eigenfrequencies of the conjugate modes. Energy pumping from the conjugate modes into axisymmetric motions takes place in the case of the second internal resonance 1:2. Analysis of the forced oscillations (traveling and standing waves) reduces to the system of six modulation equations due to the multiple scales method. The frequency response of standing and traveling waves are hard, which explained by the addition internal resonance 1:2. Moreover, the sequence of periodic doubling bifurcations of limit cycle is discovered in the case of traveling waves.

Determination of nonlinear periodic free oscillations of cylindrical shell can be carried out by nonlinear normal modes method. However, bifurcations and stability of the free oscillations of cylindrical shell can be studied by the multiple scales method. Moreover, multiple scales method is very effective for analysis of cylindrical shell forced oscillations.

Appendix

The parameters of model for free shell oscillations are the following:

$$2\chi = \frac{3}{2}\left(\frac{n^2}{2R}\right)^2; \quad \gamma = -\frac{Er^4}{8\rho} f_{10} f_{20};$$

$$\omega_1^2 = \omega_0^2 - \frac{Er^4}{16\rho}\left(f_{10}^2 - f_{20}^2\right);$$

$$\omega_2^2 = \omega_0^2 + \frac{Er^4}{16\rho}\left(f_{10}^2 - f_{20}^2\right);$$

$$\omega_0^2 = \frac{1}{\rho}\left[\frac{D}{h}(s^2 + r^2)^2 + \frac{Er^4}{R^2(s^2 + r^2)^2}\right];$$

$$\gamma_1 = \frac{1}{\rho}\left[\frac{E}{16}r^4 + \frac{Dn^4 r^4}{hR^2} - \frac{Er^4 s^4}{(s^2 + r^2)^2}\right];$$

$$g = \frac{3E}{16\rho} n^2 r^4 s^6 \left[\frac{1}{(s^2 + r^2)^2} + \frac{1}{(s^2 + 9r^2)^2}\right];$$

$$\alpha_1 = \frac{Er^4 s^4 f_{20}}{2\rho(s^2 + r^2)^2}; \quad \alpha_2 = \frac{3Er^4 s^4 f_{10}}{4\rho(s^2 + r^2)^2};$$

$$\alpha_3 = \frac{Er^4 s^4 f_{10}}{4\rho(s^2 + r^2)^2};$$

$$\beta_1 = \frac{Er^4 s^4 f_{20}}{4\rho(s^2 + r^2)^2}; \quad \beta_2 = \frac{3Er^4 s^4 f_{20}}{4\rho(s^2 + r^2)^2};$$

$$\beta_3 = \frac{Er^4 s^4 f_{10}}{2\rho(s^2 + r^2)^2}.$$

The parameters of models for forced oscillations have the following form:

$$\omega^2 = \frac{1}{\rho}\left[\frac{D}{h}(r^2 + s^2)^2 + \frac{Er^4}{R^2(r^2 + s^2)^2}\right];$$

$$\omega_3^2 = \frac{16Dr^4}{3\rho h} + \frac{E}{\rho R^2};$$

$$\gamma_1 = -\frac{Es^2}{\rho R}\left[\frac{2r^4}{(r^2 + s^2)^2} - \frac{1}{4}\right];$$

$$\gamma_2 = \frac{E}{16\rho}(r^4 + 3s^4);$$

$$\gamma_3 = \frac{Er^4 s^4}{\rho}\left[\frac{1}{(s^2 + r^2)^2} + \frac{1}{(s^2 + 9r^2)^2}\right];$$

$$\bar{\gamma}_1 = -\frac{3n^2(1 - \nu^2)[8r^4 - (r^2 + s^2)^2]}{R^2 h^2 (r^2 + s^2)^4 + 12(1 - \nu^2) r^4};$$

$$\bar{\gamma}_2 = \frac{3(1 - \nu^2)(r^4 + 3s^4) h R^3 (r^2 + s^2)^2}{4[h^2 R^2 (r^2 + s^2)^4 + 12(1 - \nu^2) r^4]};$$

$$\bar{\gamma}_3 = \frac{12(1 - \nu^2) r^4 s^4 h R^3 [(s^2 + 9r^2)^2 + (s^2 + r^2)^2]}{(s^2 + 9r^2)^2 [h^2 R^2 (r^2 + s^2)^4 + 12 r^4 (1 - \nu^2)]}.$$

The parameters for stability analysis can be presented as

$$B_{12} = -\frac{\bar{\gamma}_1}{2} a_1 a_3 (-1)^n + \frac{\bar{\alpha}_1}{2}(-1)^m;$$

$$B_{14} = -\frac{\bar{\gamma}_1}{4} a_1 a_3 (-1)^n; \; B_{41} = \frac{\bar{\gamma}_3}{6} a_1 + \frac{\bar{\gamma}_1 a_1}{12 a_3}(-1)^n;$$

$$B_{43} = -\frac{\bar{\gamma}_1 a_1^2}{24 a_3^2}(-1)^n;$$

$$B_{21} = -\frac{3}{4}\bar{\gamma}_2 a_1 - \frac{\bar{\alpha}_1}{2a_1^2}(-1)^m;$$

$$B_{23} = -\frac{\bar{\gamma}_3}{2} a_3 - \frac{\bar{\gamma}_1}{4}(-1)^n; \; B_{32} = \frac{\bar{\gamma}_1}{12} a_1^2 (-1)^n;$$

$$B_{34} = \frac{1}{2} B_{32};$$

$$\chi = \frac{\bar{\gamma}_2}{8} a_1^2 + \frac{\bar{\gamma}_1}{4} a_3 (-1)^n; \; A_1(t) = -\chi_2 \sin\theta_2;$$

$$A_2(t) = \bar{\gamma}_2 a_1 + \frac{\bar{\gamma}_2}{2} a_1 \cos\theta_2;$$

$$A_3(t) = \bar{\gamma}_3 a_3 + 0.5\bar{\gamma}_1(-1)^n \cos\theta_2;$$

$$A_4(t) = -0.5\bar{\gamma}_2 a_1^2 \sin\theta_2;$$

$$A_5(t) = 0.5\bar{\gamma}_1 a_3(-1)^n \sin\theta_2;$$

$$\chi_2 = 0.25\bar{\gamma}_2 a_1^2 + 0.5\bar{\gamma}_1 a_3(-1)^n;$$

$$\alpha_1 = \bar{\gamma}_2 a_1; \quad \alpha_2 = \bar{\gamma}_3 a_3; \quad \alpha_3 = \frac{\bar{\gamma}_2}{2} a_1;$$

$$\alpha_4 = \frac{\bar{\gamma}_1}{2}(-1)^n; \quad \alpha_5 = -\frac{\bar{\gamma}_2}{2} a_1^2;$$

$$\alpha_6 = \frac{\bar{\gamma}_1}{2} a_3(-1)^n.$$

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