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TWO-POINT PADÉ APPROXIMANTS AND THEIR APPLICATIONS TO IN SOLVING MECHANICAL PROBLEMS

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This review article deals with *two-point Padé approximants* (TPPAs) and their applications to mechanics. The attention is paid to new applications of TPPAs such as: 1) Laplace transform inversion, 2) Matching of quasilinear and essentially nonlinear asymptotics, 3) Matching of expansions for high- and low-frequency oscillations, 4) Matching of limit asymptotics in the homogenization problems, 5) TPPAs in the theory of composite materials, 6) TPPAs in the theory of nonlinear vibrations. The article deals also with one-point *Padé approximants* (PAs) and *quasifractional approximants* (QAs), which give possibility to overcome some TPPAs shortcomings.

Key words: Padé approximants, asymptotic expansions, homogenization

1. Introduction

Practically, any physical or mechanical problem, parameters of which include the non-dimensional parameter ε, can be approximately solved as ε approaches zero, or infinity. How can this kind of information be used in the study of a system at intermediate values of ε? This problem is one of the most complicated in the asymptotic analysis. As yet there has been no general answer to this tricky question of how far the parameter ε can be considered small (or large) for the investigated systems. In many instances the answer to it is given by PAs or TPPAs, which extend significantly the area of applicability of the perturbation series (Aziz and Lunardini, 1993; Baker, 1975; Baker and Graves-Morris, 1981, 1996; Brezinski and Redivo Zaglia, 1991; Bultheel, 1987; Cheney, 1988; Lorentzen and Waadeland, 1992; Dadfar and Geer, 1987; Pozzi, 1994; Vinogradov et al., 1987). Recently the PAs have been applied to asymptotic studies of the mechanical problems such as:

- Solutions to the mixed boundary value problem (Andrianov and Ivankov, 1987, 1988, 1993). The parameter ε is introduced into the boundary conditions in such a way that the case $\varepsilon = 0$ corresponds to the simple boundary problem, while $\varepsilon = 1$ corresponds to the problem under consideration.Then the ε-expansion of the solution is obtained. As a rule, at $\varepsilon = 1$ the expansion of the solution is divergent, and the PAs are used to remove this divergence. Various problems of dynamics and stability of plates and shells have been solved by using the PAs method.

- Estimations of the convergence domain for perturbation series. Practically, such estimations may be obtained by comparing perturbation series with the PAs. This approach is justified by many practical examples investigated in nonlinear mechanics (Andrianov and Bulanova, 1987; Obraztsov et al., 1991).

- Elimination of nonuniformities of asymptotic expansions. The PAs eliminates nonuniformities of asymptotic expansions in a much simpler way than, for instance well known Lighthill's method (Andrianov, 1984).

- An important property of the trigonometric PAs applied to the Fourier series, has been noted by Semerdjiev (1979). By applying the Padé transformation to the Fourier series, he observed that Gibb's phenomenon significantly diminishes (see also Nemeth and Paris, 1985).

Besides the PAs methods mentioned above there exist other approaches to the matching of asymptotic expansions. Among them the two-point Padé method is the most perspective, that is why we deal with it now.

2. Definition of two-point Padé approximants

First we give some definitions. The notion of TPPAs is defined by Baker and Graves-Morris (1981, 1996). Let

$$F(\varepsilon) = \begin{cases} \sum\limits_{i=0}^{\infty} a_i \varepsilon^i & \text{when} \quad \varepsilon \to 0 \\ \sum\limits_{i=0}^{\infty} b_i \varepsilon^{-i} & \text{when} \quad \varepsilon \to \infty \end{cases} \tag{2.1}$$

The TPPA is represented by the rational function

$$F(\varepsilon) = \frac{\sum\limits_{k=0}^{m} a_k \varepsilon^k}{\sum\limits_{k=0}^{n} b_k \varepsilon^k}$$

where $k+1$ $(k = 0, 1, 2, ..., n + m + 1)$ coefficients of a Taylor expansion, if $\varepsilon \to 0$, and $m + n + 1 - k$ coefficients of a Laurent series, if $\varepsilon \to \infty$, coincide with the corresponding coefficients of the series (2.1). Properties of the TPPAs were investigated by Draux (1991), McCabe (1975). The *two-point continued fractions* (TPCFs) are closely connected with this subject (Achutan and Ponnuswamy, 1991; Gonźales-Vera and Orive, 1994; McCabe and Murphy, 1976; Sidi, 1980b). For a heuristic role of TPPAs see Andrianov (1991b, 1993), Andrianov and Manevich (1992).

3. Simple examples

Let us investigate a model problem of vibrations of a chain consisting of n masses m, joined with springs of rigidity a. A finite difference approximation stands for a model of the longitudinal vibrations of a rod. The deflection y of the kth particle is determined by

$$m\ddot{y} = a[(y_{k+1} - y_k) - (y_k - y_{k-1})] \qquad k = 1, 2, ..., n$$

At the ends of the chain the boundary conditions are given

$$y_k = 0 \qquad \text{for} \qquad k < 1 \ \wedge \ k > n$$

There are possible:

- n proper forms of vibrations

$$y_k = A_s \sin \frac{ks\pi}{n+1} \cos(w_s t + f_s) \qquad s = 1, 2, ...n$$

– the discrete frequencies of free vibrations

$$w_s^* = 2\sqrt{\frac{a}{m}} \sin \frac{0.5 s\pi}{n+1} \tag{3.1}$$

Let us construct the asymptotic expansions of the frequency w_s^* in the vicinities of the points $s = 0$ and $s = 2(n+1)$, respectively. We introduce new variables

$$\bar{x} = \frac{x}{0.5\pi - x} \qquad x = \frac{0.5 s\pi}{n+1}$$

Thus instead of the segment $[0.2(n+1)]$ for s, we obtain the semi-infinite interval $\bar{x} \in [0, \infty)$. The expansions as $\bar{x} \to 0$ and $\bar{x} \to \infty$ take the forms

$$\sin \frac{0.5\pi\bar{x}}{1+\bar{x}} = \begin{cases} 0.5\pi\bar{x} - \bar{x}^2 + \left(1 - \frac{\pi^2}{12}\right)\bar{x}^3 - \left(1 - \frac{\pi^2}{8}\right)\bar{x}^4 + \ldots & \text{for } \bar{x} \to 0 \\ 1 - \frac{\pi^2 \bar{x}^{-2}}{8} + \left(1 - \frac{\pi^2}{12}\right)\bar{x}^{-3} - \left(1 - \frac{\pi^2}{8}\right)\bar{x}^{-4} + \ldots & \text{for } \bar{x} \to \infty \end{cases} \tag{3.2}$$

A solution obtained with the TPPA method, valid for $0 \le \bar{x} < \infty$, is given by

$$w^* = 2\sqrt{\frac{a}{m}} \frac{1.57\bar{x} + 0.81\bar{x}^2}{1 + 1.57\bar{x} + 0.81\bar{x}^2} \tag{3.3}$$

The results of the calculations of the frequency w^* are presented in Fig.1. The exact solution (3.1) is depicted by I, the expansions (3.2) by II and III. The rearranged TPPAs solution (3.3) practically coincides with the exact solution.

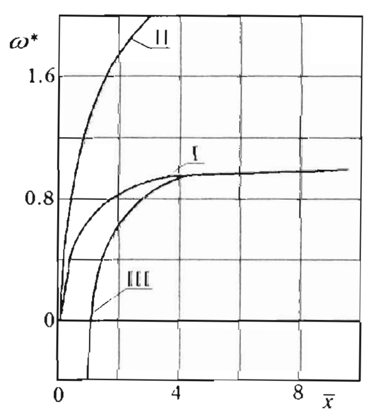

Fig. 1. Frequencies of a chain obtained by various approaches

Another interesting example is the Van der Pol equation (Andersen et al., 1984; Andersen and Geer, 1982; Andrianov and Bulanova, 1984). We give some necessary preliminary information according to Hinch (1991). The Van der Pol oscillator is governed by the equation

$$\ddot{x} + k\dot{x}(x^2 - 1) + x = 0$$

After long time the frequency and amplitude of the oscillations do not depend on the initial conditions. The limit period T is plotted in Fig.2 as a function of the coefficient of the nonlinear friction k. The curve 3 gives the numerical results obtained be means of the Runge-Kutta method. The curves 1, 2 give the second order perturbation approximations

$$T = \begin{cases} 2\pi\left(1 + \frac{k^2}{16} + O(k^4)\right) & \text{as} \quad k \to 0 \\ k(3 - 2ln2) + 7.0143k^{-1/3} + O(k^{-1}lnk) & \text{as} \quad k \to \infty \end{cases} \tag{3.4}$$

The TPPA formula constructed from two terms of the expansion $(3.4)_1$ and one term of the expansion $(3.4)_2$

$$T = \frac{6.2832 + 1.5294k + 0.3927k^2}{1 + 0.2433k}$$

shows a good agreement with the numerical results for all values of $k > 0$ (curve 4 in Fig.2).

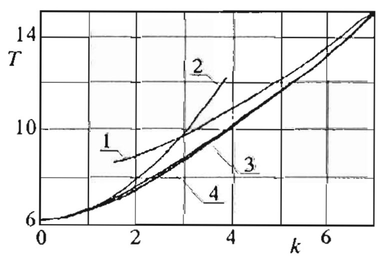

Fig. 2. Period of the Van der Pol pendulum: comparison of numerical, perturbative and TPPA solutions

4. Using of the TPPAs

4.1. Problem of hydrofoil

Let us consider the problem of a hydrofoil. For great values of a relative submerging h, the relative hydrodynamic lift Q of a thin plate is expressed as follows (Panchenkov, 1976)

$$Q = 1 - \frac{1}{16h^2} + O(h^{-3}) \tag{4.1}$$

For $h \to 0$ (hydroplaning), we have

$$Q = 0.5 \tag{4.2}$$

The exact solution for intermediate values of the parameter h is not known. In the monograph of Panchenkov (1976), it is proposed to use the method of functional parameters. However this approach is on the contrary, the formulae (4.1) and (4.2) allow us to construct quite simply cumbersome the TPPAs, which give the values of G for any h

$$Q = \frac{16h^2 + 1}{16h^2 + 2} \tag{4.3}$$

Fig.3 presents the numerical values calculated from Eqs (4.1) and (4.3) (curves 2 and 1, respectively) and the experimental data taken from Panchenkov's monograph (1976) (curve 3). The correspondence between experimental data and the results of the TPPA formula is quite satisfactory.

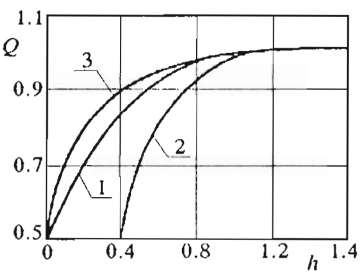

Fig. 3. Using of the asymptotic and TPPA approaches when solving the problem of hydrofoil

4.2. Inverse Laplace transform problem

Papers of Andrianov (1992), Grundy (1977) were devoted to the following problem. Let us consider the Laplace transform

$$F(p) = 0.5\sqrt{p}[H_0(p) - Y_0(p)]$$

where H_0 – Struve function, Y_0 – Bessel function, p – parameter of the Laplace transform.

The exact inverse is

$$f(t) = \sqrt{1+t^2}$$

The asymptotic inverses take forms

$$f(t) \simeq \begin{cases} 1 - 0.5t^2 + \ldots & \text{for} \quad t \to 0 \\ t^{-1} + \ldots & \text{for} \quad t \to \infty \end{cases}$$

By using the TPPAs one obtains

$$f(t) = \frac{1 + 0.5t}{1 + 0.5t + 0.5t^2} \tag{4.4}$$

The numerical results are plotted in Fig.4. The upper curve (Eq (4.4)) coincides satisfactorily with the lower one (exact solution).

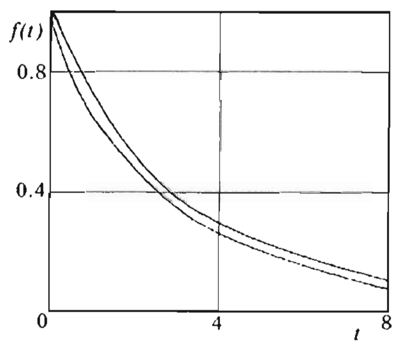

Fig. 4. Laplace transform inverse, the exact and TPPA solutions

Hence the rational function (4.4) is an "asymptotically equivalent function" in the sense of Slepyan and Yakovlev (1980). The accuracy of the TPPA solution may be improved by removing of the essential transform singularities (Krylov and Skoblya, 1974). The other approximate methods (Longman, 1973; Sidi, 1980a; Talbot, 1979; Van Iseghem, 1987) enable the error analysis. The TPPAs approach can also be applied to other integral transforms (Fourier, Bessel, Mellin and so on).

4.3. Matching of quasi-linear and essentially nonlinear asymptotics

The next example deals with the problem of oscillations of a plate on a nonlinear elastic foundation (Andrianov and Bulanova, 1995). That problem may be solved using numerical or quasilinear asymptotic methods. In the last one quasi-linear asymptotics are usually used. For large amplitudes the solutions can be obtained as follows. In the long-wave approximation the plate bending rigidity may be neglected and we can investigate oscillations of the body on the elastic nonlinear spring. In a short-wave approximation the nonlinearity of the foundation is negligible as well. For such a case the TPPAs method is very suitable. It is also suitable for solving the nonlinear problems for beams, plates and shells. Now we are going to give an example of TPPAs application to the nonlinear theory of shells (Evkin, 1989). Within the frame of the theory of buckling shells, the solution presented below has been obtained by means of the asymptotic method for a closed sphere subjected to the uniform external pressure q

$$Q = 0.42\varepsilon + 0.3\varepsilon^3 + O(\varepsilon^5) \tag{4.5}$$

Here

$$\varepsilon = 2W\sqrt{3\sqrt{1-\nu^2}} \qquad Q = \frac{0.5qR^2\sqrt{3(1-\nu^2)}}{Eh^2} \qquad W = \frac{w}{h}$$

and w – amplitude of supercritical axially-symmetric equilibrium form, E – modulus of elasticity, ν – Poisson ratio, h – shell thickness. In the region of small deflection the approach of Koiter is valid, which gives the solution expansion in the form

$$Q = 1 + O(\varepsilon^{-4}) \tag{4.6}$$

By matching the expansions (4.5) and (4.6) with the TPPA, we obtain the solution in the form

$$Q = \frac{A}{A + 2.19} \tag{4.7}$$

Here

$$A = \varepsilon^4 + 0.082\varepsilon^3 + 0.386\varepsilon^2 + 0.92\varepsilon$$

Curve 1 in the Fig.5 corresponds to the solution (4.5). The precise numerical results, obtained by Gabrilyantz and Feodos'ev (1961) (practically the exact solution) is presented by the curve 2. The curve corresponds to the solution (4.7) practically coincides with the exact solution (i.e., with the curve 2 in Fig.5).

The model of shell discussed above was used for estimation of the critical pressure for cylindrical shells with initial imperfections (Evkin and Krassovsky, 1991). Moreover, the TPPAs were also applied for matching the coefficients of the governing equations as functions of time (Stankevich et al., 1991).

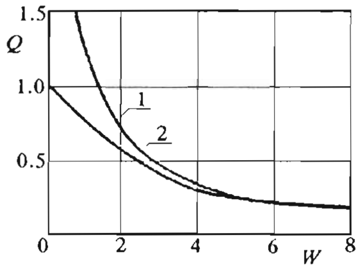

Fig. 5. Using TPPA for matching of quasilinear and essentially nonlinear asymptotics

4.4. Matching of expansion for high and low-frequency oscillations

Dynamics of a ribbed plate is described by a system of partial differential equations with discontinuous coefficients. Numerical methods are not efficient and very often are not acceptable for such equations. It is possible to use the homogenization method (Bakhvalov and Panasenko, 1989) for the low frequency case. At the first stage, rigidities and densities of lateral ribs are spread along the plate and the plate itself is replaced by a smooth orthotropic plate with effective rigidities and densities. Further on, using the Vishik-Lusternik first approximation approach, corrections to the frequencies and displacements, caused by discretization, are obtained. In the high-frequency case a perturbation method (Nayfeh, 1973) is used, and the theory of smooth plates plays the role of the first approximation. In this section the homogenization and perturbation solutions are to be matched by the TPPAs. As a result of the application of the above-mentioned method an analytic expression has been deduced which describes the oscillations of a ribbed plate on elastic foundation for the whole spectrum of frequencies. A comparison with known solutions was made and accuracy of the method was found. Let us consider the ribbed plate $(0 \le x \le L_1,\ L_2 \le y \le L_2)$ resting on the Winkler elastic foundation of the stiffness C_1. To a great extent this model represents the basic features of the real system oscillations. The model equations are given by

$$D\nabla^4 W + E_1 I P(y) W_{xxxx} + C_1 W + \lambda[\rho h + \rho_1 EFP(y)]W = 0 \qquad (4.8)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad P(y) = \sum_{i=-n}^{n} \delta(y - ib) \qquad b = \frac{2L_2}{N} \qquad n = 0.5(N-1)$$

and N – number of ribs, $N = 2k+1$, ρ – mass density of the plate material, E_1, ρ_1 – modulus of elasticity and mass density of the rib material, δ – Dirac function, λ – square of frequency, I – inertia moment, F – square of the cross-section of the rib.

The boundary conditions on the edges of the plate may be formulated as

$$\begin{aligned} &W_{yy} = W_{yyy} = 0 \quad \text{when} \quad y = \pm L_2 \\ &W_{xx} = W_{xxx} = 0 \quad \text{when} \quad x = 0 \ \wedge \ x = L_1 \end{aligned} \tag{4.9}$$

The homogenization procedure leads to the following boundary value problem

$$D\nabla^4 W_0 + E_1 I b^{-1} W_{0xxxx} + C_1 W_0 - (\rho h + \rho_1 F b^{-1})\lambda_0 W_0 = 0 \tag{4.10}$$

and the boundary conditions (4.9) (after replacing W with W_0). The approach used allows us to determine the expansions of frequencies and oscillation modes with any desired accuracy. Now we will investigate high-frequency oscillations. Let us introduce new "fast" ξ, η variables given by

$$\xi = \varepsilon^\alpha x \qquad \eta = \varepsilon^\alpha y \qquad \alpha > 0$$

Here ε is an auxiliary small parameter. Then the derivatives may be rewritten as follows

$$\frac{\partial}{\partial x} = \varepsilon^\alpha \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial y} = \varepsilon^\alpha \frac{\partial}{\partial \eta} \tag{4.11}$$

The plate oscillation mode and frequency squared asymptotic expansion may be found as

$$\begin{aligned} W &= W_1(\xi,\eta) + \varepsilon^\alpha W_2(\xi,\eta) + \ldots \\ \lambda &= \varepsilon^{-4\alpha}(\lambda_1 + \varepsilon^\alpha \lambda_2 + \ldots) \end{aligned} \tag{4.12}$$

Substituting Eqs (4.11), (4.12) into Eq (4.8) and performing the ε-splitting, from the system of equations that determines the unknown expansion coefficients, one obtains

$$D\nabla^4 W_1 - \lambda_1 \rho h W_1 = 0 \tag{4.13}$$

$$D\nabla^4 W_2 - \lambda_1 \rho h W_2 = -C_1 W_1 - E_1 I P W_{\xi\xi\xi\xi} + \lambda_2 \rho h W_1 + \lambda_1 \rho_1 E F P W_2 \tag{4.14}$$

Eq (4.13) describes the smooth plate oscillations, while Eq (4.14) allows one to obtain the frequency and mode expansions for the first order of ε. The values of the parameters used in the numerical analysis are: $N = 11$, $C_1 L_1^4/D = 0.1$, $E_1 F/(Db) = 200$, $\rho_1 F/\rho b h = 0.5$, $L_2/L_1 = 1$, $m = 1$, $0 < k < 80$. Here

$m(k)$ is the wave number in the direction of the x (y) axis. The results are plotted in Fig.6, here the curves correspond to:

I – the orthotropic model of the oscillation frequency; II – the case of the smooth plate's oscillation frequency; III and IV – the truncated series for the low- and high- frequency asymptotics' (only the first two terms of expansion are taken into account). The dotted line represents the values of the frequency determined by numerical methods. Curve V corresponds to the matched spectrum expression.

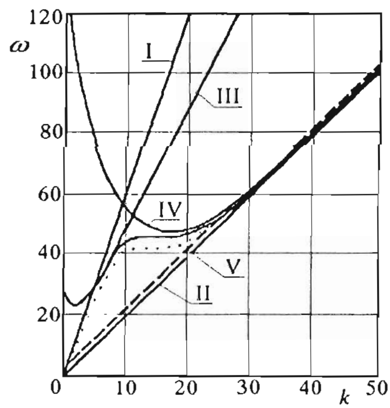

Fig. 6. Matching by the TPPA of homogenized and higher-frequency solutions

The plots in Fig.6 show that the values of frequency obtained using the approximate analytical and numerical methods lay inside the region bounded by the curves 1 and 2. This result ascertains the physical nature of the problem and confirms reliability of the solution. Furthermore, the comparison with the numerical data shows that for $0 < k < \infty$ the curve V coincides satisfactorily with the numerical solution. Thus, the TPPA method provides the closed analytical formula for the total spectrum of the plate natural frequencies (see also Obraztsov et al., 1991).

4.5. Matching of limiting asymptotics in the homogenization problem

The theory of homogenization has been developed for perforated media by many authors in recent years (see Bakhvalov and Panasenko, 1989). The main problem in this field is solving of the so called cell (or local) – boundary value problem for the periodically repeated element with conditions of periodicity.

This problem has usually been studied by means of numerical methods. For solving the cell problem the asymptotic methods were used by Andrianov and Shevchenko (1988, 1989), Andrianov et al. (1995). Let us assume the example: bending of rectangular plate with circular perforations. Analytical solutions for small and large holes were obtained by Andrianov and Shevchenko (1989) by using the asymptotic methods (perturbation of the domain and boundary form). For the coefficients A and B of the homogenized equation

$$A(W_{xxxx} + W_{yyyy}) + 2BW_{xxyy} = q(x, y)$$

we have the following expressions (for $\nu = 0.3$)

$$A = \frac{1-\varepsilon}{1 - 0.5785\varepsilon} \qquad B = \frac{1-\varepsilon}{1 - 0.6701\varepsilon}$$

where $\varepsilon = b/a$, b – diameter of the hole, a – length of the square cell side. Fig.7 shows the numerical results for A and B.

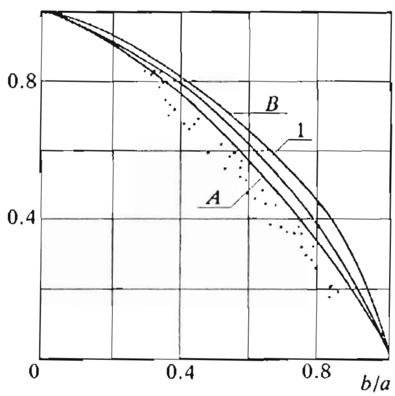

Fig. 7. Comparison of the TPPA solution for homogenized coefficients of perforated plates with other analytical approaches and experimental data

The values of coefficients are compared with the theoretical results obtained by means of the two-period elliptic functions method (Grigolyuk and Phil'shtinsky (1970), $A = B$, curve 1 on Fig.7) and experimental results (points on Fig.7, Grigolyuk and Phyl'shtinsky, 1970). The accuracy of the TPPA method is visible.

4.6. Theory of composite materials

Transport coefficients of composite materials may be evaluated effectively using the method of bounds. The bounds become increasingly accurate when more information on geometrical properties of the medium is known. For two-component isotropic composites, the PAs bounds for the effective constants λ_e/λ_1 already exist (Bergman, 1978; Milton and Golden, 1985; Milton, 1986; May et al., 1994). These bounds are usually obtained in the form of *continued fractions* (CFs) on the basis of the analytic properties of $\lambda_e(\lambda_1,\lambda_2)$. Bergman (1978) studied these analytic properties of $\lambda_e(\lambda_1,\lambda_2)$. He proved that $\lambda_e(\lambda_1,\lambda_2)/\lambda_1 = \lambda_e(1,\lambda_2/\lambda_1)$ is a Stieltjes function of λ_2/λ_1, analytic except for the negative real axes, satisfying $\lambda_e(\lambda_1,\lambda_2)/\lambda_1 > 0$ when $\lambda_2/\lambda_1 > 0$. The Stieltjes functions have been extensively studied in the mathematics literature and their PAs and CFs representations are well known (Gilewicz, 1978; Jones and Thron, 1980; Graves-Moris and Baker, 1981, 1996).

On the contrary, the analytic properties of TPPAs generated by two different power expansions of Stieltjes function have not been examined as deeply as the PAs. The authors concerned themselves mostly with the TPPAs using equal number of coefficients of two power expansions at zero and infinity ("balanced" situation).

The convergence of TPPAs has been investigated by (Jones and Thron, 1970; Gragg, 1980; Jones et al., 1983). In the paper of Casasus and González-Vera (1985) the analytical properties of a special type of TPPAs to the Stieltjes functions are examined. Monotone sequences of TPPAs forming upper and lower bounds for the Stieltjes functions have been reported by González-Vera and Njåstad (1990).

Recently (Tokarzewski et al., 1994a; Tokarzewski, 1996a,b) investigated the TPPAs to non-equal, finite number of terms of two power expansions of the Stieltjes functions at zero and infinity ("unbalanced" situation). Under some assumptions they proved that the diagonal TPPAs form sequences of lower and upper bounds uniformly converging to the Stieltjes function.

The general "unbalanced" situation, i.e., the TPPA corresponding to an arbitrary number of terms of power expansions at zero and infinity has been studied in real domain by Bultheel et al. (1995) and independently by Tokarzewski and Telega (1996b, 1997). They extended the fundamental inequalities derived for the PAs (Baker, 1975) on the general "unbalanced" TPPAs case. They proved the following theorem very useful for practical applications.

Theorem 1. The TPPAs to the Stieltjes function $R(x) = \int_0^\infty d\gamma(u)/(1+x)$ represented by the following power expansions at zero

$$R(x) \simeq \sum_{n=1}^{\infty} c_n x^n$$

and infinity

$$R(x) \simeq \sum_{n=0}^{\infty} C_{-n} x^{-n} \qquad \left(R(x) \simeq C_1 x + \sum_{n=-1}^{\infty} C_{-n} x^{-n}\right)$$

obey for $k = 1, 2, \ldots, 2M$ $(k = 1, 2, \ldots, 2M + 1)$ the following inequalities

$$(-1)^k [M/M]_k < (-1)^k [(M+1)/(M+1)]_k < (-1)^k R(x)$$
$$\left((-1)^{k-1} [M/(M-1)]_k < (-1)^{k-1} [(M+1)/M]_k < (-1)^{k-1} R(x)\right)$$

where $R(x)$ stands for the limit as M tends to infinity of $[M/M]_k$, $\left([(M+1)/M]_k\right)$, and x is real and positive. Note that we introduced the following notation of TPPA, cf. Section 3

$$[M/N]_k = \frac{\sum_{j=0}^{M} \alpha_j x^j}{\sum_{j=0}^{N} \beta_j x^j}$$

Here k denotes the given number of coefficients of power expansions at infinity matched by the TPPA represented by $[M/N]_k$. The above inequalities have the consequence that $[M/M]_k$ and $[(M+1)/M]_k$ form upper and lower bounds on $R(x)$ obtainable using only the given number of coefficients, and that the use of additional coefficients (higher M) improves the bounds.

Theorem 1 have been successfully used (Tokarzewski et al., 1994b) for the study of the effective heat conductivity $\lambda_e(h)$ $(h = \lambda_1/\lambda_2)$ for a periodic square array of the cylinders of conductivity $\lambda_2 = h$ $(\lambda_2 = 1)$ embedded in the matrix of conductivity $\lambda_1 = 1$ $(\lambda_1 = h)$. As an input for calculation of TPPAs the authors used coefficients of the expansions of $\lambda_e(x)$ in powers of x and in powers of $1/x$, where $x = h - 1$, $h = \lambda_2/\lambda_1$. The sequences of TPPAs uniformly converging to the effective conductivity $\lambda_e(h)$ are shown in Fig.8. The best bounds obtained by the TPPAs method, namely $[18/18]_1$ and $[18/18]_2$, are presented in Fig.9. In all Fig.8 and Fig.9 the asymptotic solution obtained by McPhedran et al. (1988) is drawn for comparison.

It follows that the TPPAs approach allow us to evaluate the effective moduli for a range of parameters much larger then the PAs methods reported in literature (McPhedran and Milton, 1981; May et al., 1994). For example for

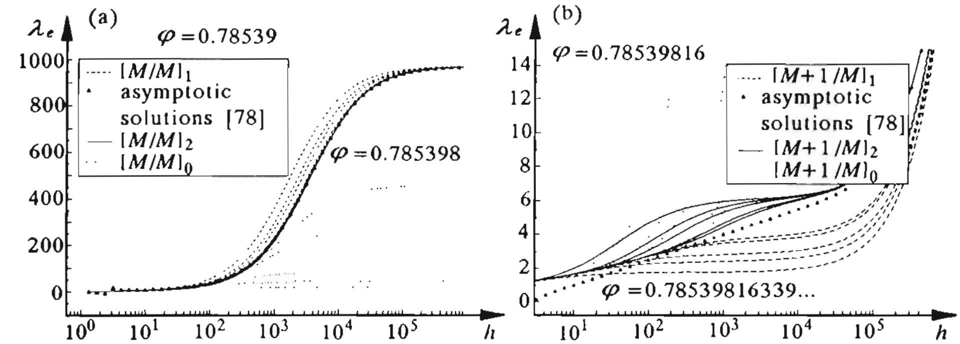

Fig. 8. The sequences of $[M/M]_0$, $[M/M]_1$ and $[M/M]_2$, $M = 2, 4, 6, 12, 18$ uniformly converging to the effective conductivity $\lambda_e(h)$ $(h = \lambda_2/\lambda_1)$ of square array of cylinders. The curves $[M/M]_2$ are indistinguishable (solid line – (a)). The bounds $[18/18]_1$ and $[18/18]_2$ are very restrictive

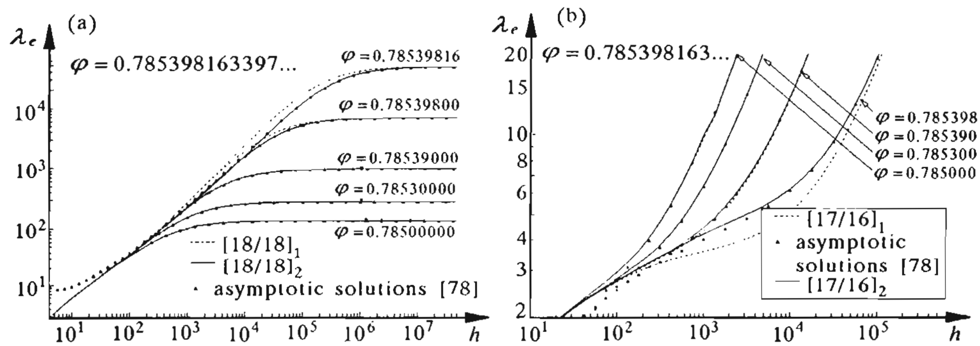

Fig. 9. The TPPA upper and lower bounds on the effective conductivity for a square array of densely packed highly conducting cylinders. For $\varphi = 0.785$ the bounds coincide. For $\varphi = 0.7853, 0.78539$ are very restrictive. For higher volume fractions $\varphi \geq 0.78539816$ the difference between lower and upper bounds increases rapidly

$\varphi = 0.78539$ the TPPAs approach leads to very restrictive bounds, whereas the PAs methods fails, Fig.8a and Fig.9a.

The TPPA method presented in this Section can be applied to calculation of the bounds on dielectric constants, magnetic permeabilities, viscous coefficients, elastic constants and others.

For some special input data also three- and four-point PAs were used for estimation of the effective conductivity of two-component medium (Helsing, 1993, 1994).

4.7. Matching of local expansions in the theory of nonlinear vibrations

Interesting results were obtained using the PA method in the theory of normal vibrations of nonlinear finite-dimensional systems. Normal vibrations in the nonlinear case are generalization of corresponding normal vibrations of linear systems. In the normal mode a finite-dimensional system behaves like a conservative one having a single degree of freedom. In this case all position coordinates can be well defined from any one of them by

$$x_i = p_i(x) \qquad x \equiv x_1 \qquad i = 2, 3, ..., n \tag{4.15}$$

with $p_i(x)$ being analytical functions. Rosenberg and Atkinson (1959), Rosenberg (1966) get credit for being the first to introduce broad classes of essentially nonlinear conservative systems allowing normal vibrations with rectilinear trajectories in the configurational space

$$x_i = k_i x_1 \qquad i = 2, 3, ..., n \tag{4.16}$$

For instance, the homogeneous system potential of which is an even homogeneous function of the variables refer to such a class. It is interesting to note that the number of modes of normal vibrations in the nonlinear case can exceed the number of degrees of freedom. This remarkable property has no analogy in the linear (non-degenerate) case. In systems of a more general type, trajectories of normal vibrations are curvilinear. Lyapunov (1992) showed that solutions of this kind exist in nonlinear finite-dimensional systems with an analytical first integral which are close to generating linear systems. New results concerning normal vibrations with curvilinear trajectories in essentially nonlinear case have been obtained by Manevich and Mikhlin (1972, 1989), Mikhlin (1985), Manevich et al. (1989).

Consider a conservative system

$$\begin{gathered} m_i \ddot{x}_i + \Pi_{x_i} = 0 \\ \dot{x}_i = \frac{dx_i}{dt} \qquad \Pi_z = \frac{\partial \Pi}{\partial z} \qquad i = 1, 2, ..., n \end{gathered} \tag{4.17}$$

$\Pi = \Pi(\boldsymbol{x})$ being potential energy. Π is assumed to be a positively definite function; $\boldsymbol{x} = [x_1, x_2, ..., x_n]^\top$. Power series expansion for $\Pi(\boldsymbol{x})$ begins with the terms having a power of at least 2. Without reducing the degree of generalization, assume that $m_i = 1$ since this can be always ensured by dilatation of coordinates. The energy integral for system (4.17) is

$$\frac{1}{2}\sum_{k=1}^{n} x_k^2 + \Pi(x_1, x_2, ...x_n) = h \tag{4.18}$$

h being the system energy. Assume that within the configuration space, bounded by a closed maximum equipotential surface $\Pi = h$, the only equilibrium position is $x_i = 0$ $(i = 1, 2, ..., n)$. In order to determine the trajectories of normal vibrations (4.15), the following relationships can be used

$$2x_i'' \frac{h - \Pi}{1 + \sum_{k=2}^{n} x_k'^2} + x_i'(\Pi_x) = -\Pi_{x_i} \qquad i = 2, 3, ..., n \quad x \equiv x_i \tag{4.19}$$

These are obtained either as the Euler equations for the variational principle in the Jacobi form or by elimination of time from the equations of motion (4.17) with consideration of the energy integral (4.18). An analytical extension of the trajectories on a maximum isoenergetic surface $\Pi = h$ is possible if the boundary conditions, i.e. the conditions of orthogonality of a trajectory to the surface, are satisfied (Rosenberg, 1966)

$$x_i'\Big[-\Pi_x(X, x_2(X), ..., x_n(X))\Big] = -\Pi_{x_i}\Big(X, x_2(X), ...x_n(X)\Big) \tag{4.20}$$

$\Big(X, x_2(X), ...x_n(X)\Big)$ being the trajectory return points lying on $\Pi = h$ surface where all velocities are equal to zero. If a trajectory $x_i(x)$ is defined, the law of motion with respect to time can be found using

$$\ddot{x} + \Pi_x\Big(x, x_2(x), ..., x_n(x)\Big) = 0$$

for which a periodic solution $x(t)$ is obtained by inversion of the integral. Manevich and Mikhlin (1972) developed nearly rectilinear trajectories of normal vibrations in the form of power series. Now consider a problem of normal vibrational behavior in some nonlinear systems when the amplitude (or energy) of the vibrations is varied from zero to an extremely large value. Assume that in a system

$$\ddot{z} + \Pi_{z_i}(z_1, z_2, ..., z_n) = 0 \tag{4.21}$$

the potential energy $\Pi(z_1, z_2, ..., z_n)$ is a positive definite polynomial of $z_1, ..., z_n$ having a minimum order of two and a maximum of $2m$. On choosing a coordinate, say z_1, substitute $z_i = cx_i$ where $c = z_1(0)$. Obviously, $x_1(0) = 1$. Furthermore, without loss of generality assume $\dot{x}_1 = 0$. Eqs (4.21) can be rewritten

$$\ddot{x}_i + V_{x_i}(c, x_1, x_2, ..., x_n) = 0 \tag{4.22}$$

where

$$V = \sum_{k=0}^{2m-2} c^k V^{(k+2)}(x_1, x_2, ..., x_n)$$

$V^{(r+1)}$ contains terms of the power $(r+1)$ of the variables in the potential

$$V(c, x_1, x_2, ..., x_n) \equiv \Pi\left(z_1(x_1), z_2(x_2), ..., z_n(x_n)\right)$$

It is assumed below that the amplitude of vibrations $c = z(0)$ is an independent parameter. At small amplitudes a homogeneous linear system with a potential energy $V^{(2)}$ is selected as the initial one while at large amplitudes, a homogeneous nonlinear system with a potential energy $V^{(2m)}$ is selected. Both linear and nonlinear homogeneous systems allow normal vibrations of the $x_i = k_i x_1$ type, where the constants k_i are determined from the algebraic equations

$$k_i V_{x_1}^{(r)}(1, k_2, ..., k_n) = V_{x_i}^{(r)}(1, k_2, ..., k_n)$$

(A number of possible vibrations of this type can be greater than the number of degrees of freedom in the nonlinear case). In the vicinity of a linear system at small values of c the trajectories of normal vibrations $x_i^{(1)}(x)$ can be determined as power series of x and c (assuming that $x_1 \equiv x$, while in the vicinity of a homogeneous nonlinear system (at large values of c), $x_i^{(2)}(x)$, as power series of x and c^{-1}. Construction of the series is described by Manevich et al. (1989). The amplitude values (at $\dot{x} = \dot{x}_i = 0$) define the normal vibration modes completely. Therefore, for the sake of simplicity, only expansions of $\rho_i^{(1)} = x_i^{(1)}(1)$ and $\rho_i^{(2)} = x_i(1)$ in terms of powers of c will be discussed below

$$\rho_i^{(1)} = \sum_{j=0}^{\infty} \alpha_j^{(i)} c^j \qquad \rho_i^{(2)} = \sum_{j=0}^{\infty} \beta_j^{(i)} c^{-j} \tag{4.23}$$

In order to join together the local expansions (4.23) and investigate behavior of normal vibration the method of TPPA were used (Mikhlin, 1985, 1995; Vakakis et al., 1996). We obtain

$$P_s^{(i)} = \frac{\sum_{j=0}^{s} a_j^{(i)} c^j}{\sum_{j=0}^{s} b_j^{(i)} c^j} \qquad s = 1, 2, 3, ... \quad i = 2, 3, ..., n \tag{4.24}$$

or

$$P_s^{(i)} = \frac{\sum_{j=0}^{s} a_j^{(i)} c^{j-s}}{\sum_{j=0}^{s} b_j^{(i)} c^{j-s}} \qquad s = 1, 2, 3, ... \quad i = 2, 3, ..., n \tag{4.25}$$

Compare Eqs (4.24) and (4.25) with Eq (4.23). By preserving only the terms with an order of c^r $(-s \le r \le s)$ and equating the coefficients at equal powers of c, $n-1$ systems of $2(s+1)$ linear algebraic equations will be obtained for determination of a_j^i, $b_j^{(i)}$ $(j = 0,1,2,...)$. Since the determinants of these systems $\Delta_s^{(i)}$ are generally not equal to zero, the systems of algebraic equations have a single exact solution, $a_j^{(i)} = b_j^{(i)} = 0$. Select the TPPA corresponding to the preserved terms in Eq (4.23) having nonzero coefficients $a_j^{(1)}$, $b_j^{(1)}$. Assume that $b_0^{(i)} \neq 0$, for otherwise at $c \to 0$, $x_i^{(1)} \to \infty$. Without loss of generality it can also be assumed that $b_0^{(i)} = 1$. Now the systems of algebraic equations for determining of $a_j^{(i)}$, $b_j^{(i)}$ become overdetermined. All unknown coefficients $a_0^{(i)}$, $a_n^{(i)}$, $b_1^{(i)}$, $b_n^{(i)}$, $i = 2,3,...,n$ are determined from $(2s+1)$ equations while the "error" of this approximate solution can be obtained by substituting all coefficients in the remaining equation. Obviously, the "error" is determined by the value of $\Delta_s^{(i)}$ since at $\Delta_s^{(i)} = 0$ nonzero solutions and consequently, the exact PA will be obtained in the given approximation in terms of c. Hence the following condition

$$\Delta_s^{(i)} \to 0 \wedge \text{for} \wedge s \to \infty \wedge i = 2,3,...,n \tag{4.26}$$

is necessary for convergence of a sequence of TPPAs (4.24) to the rational function

$$P^{(i)} = \frac{\sum\limits_{j=0}^{\infty} a_j^{(i)} c^j}{\sum\limits_{j=0}^{\infty} b_j^{(i)} c^j} \qquad (b_0^{(i)} = 1) \tag{4.27}$$

Indeed, if the conditions (4.26) are not satisfied, nonzero values of the coefficients $a_j^{(i)}$, $b_j^{(i)}$ in Eq (4.27) will obviously not be obtained. The conditions (4.26) are necessary but not sufficient for the convergence of approximants (4.24) to functions (4.27); nevertheless, the role of conditions (4.26) can be clarified as follows. In the general case there are more than one quasilinear local expansions and essentially nonlinear local expansions alike, the numbers of expansions of the respective type being not necessarily equal, it is the convergence conditions (4.26) that allow one to establish a relation between the quasilinear and essentially nonlinear expansions, that is to decide which of them corresponds to the same solution and which to different ones. For a concrete analysis based on the above technique, consider a conservative system with two degrees of freedom, potential energy of which contains terms of 2nd and 4th powers of variables z_1, z_2. Substituting $z_1 = cx$, $z_2 = cy$, where $c = z_1(0)$ for $x(0) = 1$, one obtains

$$V = c^2\left(d_1\frac{x^2}{2} + d_2\frac{y^2}{2} + d_3xy\right) +$$

$$+ \quad c^4\Big(\gamma_1\frac{x^4}{4}+\gamma_2x^3y+\gamma_3\frac{x^2y^2}{2}+\gamma_4xy^3+\gamma_5\frac{y^4}{4}\Big) \equiv c^2V^{(2)}+c^4V^{(4)}$$

The equation representing the trajectory $y(x)$ is of the form

$$2y''(h-V)+(1+y^2)(-y'V_x+V_y)=0 \tag{4.28}$$

while the boundary conditions (4.20) can be written on the isoenergy surface $h=V$

$$(-y'V_x+V_y)=0$$

For definiteness, let

$$\begin{array}{lll} d_1=d_2=1+\gamma & d_3=-\gamma & \\ \gamma_1=1 & \gamma_2=0 & \gamma_3=3 \\ \gamma_4=0.2091 & \gamma_5=\gamma & \end{array}$$

Write equations of motion for the following system

$$\begin{aligned} &\ddot{x}+x+\gamma(x-y)+c^2(x^3+3xy^2+0.2091y^3)=0 \\ &\ddot{y}+y+\gamma(y-x)+c^2(2y^3+3x^2y+0.6273y^2x)=0 \end{aligned} \tag{4.29}$$

In the linear limiting case $(c=0)$ two rectilinear normal modes of vibrations $y=k_0x$, $k_0^{(1)}=1$, $k_0^{(2)}=-1$ are obtained,while a nonlinear system (equations of motion contain only third power terms with respect to x, y) admits four such modes: $k_0^{(3)}=1.496$, $k_0^{(4)}=0$, $k_0^{(5)}=-1.279$, $k_0^{(6)}=-5$. In order to determine nearly rectilinear curvilinear trajectories of normal vibrations, Eq (4.28) is used along with the boundary conditions. By matching of the local expansions the following PA are obtained

$$\begin{array}{lll} I-IV & \gamma=2: & \rho=\dfrac{1+1.20c^2}{1+1.61c^2+0.72c^4} \\ & \gamma=0.5: & \rho=\dfrac{1+1.06c^2}{1+2.06c^2+3.20c^4} \\ & \gamma=0.2: & \rho=\dfrac{1+1.70c^2}{1+3.96c^2+13.29c^4} \\ II-V & \gamma=2: & \rho=\dfrac{-1-1.11c^2-0.275c^4}{1+1.00c^2+0.215c^4} \\ & \gamma=0.5: & \rho=\dfrac{-1-2.76c^2-1.36c^4}{1+2.31c^2+1.04c^4} \\ & \gamma=0.2: & \rho=\dfrac{-1-6.41c^2-9.03c^4}{1+5.30c^2+7.02c^4} \end{array} \tag{4.30}$$

The two additional modes of vibration exist only in a nonlinear system; as $\nu = c^2$ increases (amplitude c decreases), they vanish at a certain limiting point. For the analysis of these vibration modes, assume a new variable $\sigma = (\rho - 1.496)/(\rho - 5)$. By using the variable σ two expansions in terms of positive and negative powers were obtained; therefore, fractional rational representations can be introduced as above. By comparing these expansions, the following TPPAs are obtained

$$\begin{aligned}
\gamma = 2 : \qquad & \nu = \frac{8.874\sigma + 1.126\sigma^2}{1 + 4.300\sigma + 2.836\sigma^2 + 0.549\sigma^3} \\
\gamma = 0.5 : \qquad & \nu = \frac{35.497\sigma + 5.108\sigma^2}{1 + 3.021\sigma - 0.794\sigma^2 + 0.622\sigma^3} \\
\gamma = 0.5 : \qquad & \nu = \frac{88.986\sigma + 1.470\sigma^2}{1 - 0.143\sigma + 3.747\sigma^2 + 0.072\sigma^3 F^4}
\end{aligned} \tag{4.31}$$

Now proceed to the determination of the limiting point. Obviously, it can be found from $\partial\nu/\partial\sigma = 0$. From Eqs (4.31) at $\gamma = 2$ the limiting point is $\nu \approx 1.21$, $c \approx 0.91$; at $\gamma = 0.5 - \nu \approx 11.10$, $c \approx 0.30$; at $\gamma = 0.2 - \nu \approx 23.93$, $c \approx 0.20$. Hence, as $\gamma \to 0$ the limiting point is characterized by the amplitude $c \to 0$. Therefore, the two additional vibration modes in a nonlinear system can exist at rather small amplitudes of vibrations. Note that quasilinear analysis does not allow one to find these solutions even at small amplitudes. In the limit, when $\gamma = 0$, a linear system decomposes into two independent oscillators of identical frequencies and admits any rectilinear modes of normal vibrations. Obviously, a full system (4.29) at $\gamma = 0$ admits four modes of vibrations (in a nonlinear case) $y_2 = ky_1$, $k = 1.496, 0, -1.279, -5$. Thus, the TPPAs allow of judgment of nonlocal behavior of normal vibrations in nonlinear finite-dimensional systems. For the system (4.29) the evolution of modes of normal vibrations is shown in Fig.10 using parameters $\zeta = Ln(1 + c^2h^2)$ and $\varphi = \arctan\rho$ (the picture shows periodicity in φ, the period being 2π). Solid lines correspond to analytical solutions (Eqs (4.30) and (4.31) were employed) while the dotted ones were obtained in computer check calculations at $\gamma = 2$ carried out by Zhupiev. The analytical solutions and numerical calculations show good agreement. For the solution II $\div$ V, relationship (4.30) and the numerical calculation gave, in the scale selected, the same curve (Fig.10).

Note that the systems considered above can be obtained in calculations of nonlinear vibrations of shells (using the Bubnov-Galerkin technique) as well as in other problems. For instance, in the problem of vibrations of a construction on elastic supports under a force having a constant direction.

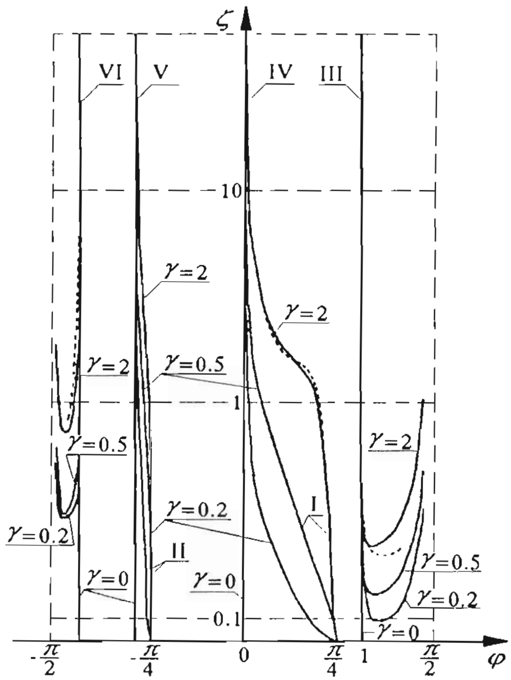

Fig. 10. Matching asymptotic expansions in nonlinear normal modes for arbitrary energies

5. Quasifractional approximants

Evidently, the TPPAs is not a panacea. For example, one of the "bottlenecks" of the TPPAs method is related to the presence of logarithmic components in numerous asymptotic expansions. Van Dyke (1975) wrote: "A technique analogous to rational functions is needed to improve the utility of series containing logarithmic terms. No striking results have yet been achieved. We give an example of partial success". This problem is the most essential for the TPPAs, because, as a rule, one of the limits ($\varepsilon \to 0$ or $\varepsilon \to \infty$) for a real mechanical problems gives expansions with logarithmic terms or other complicated functions. It is worth noting that in some cases these obstacles may be overcome by using an approximate method of TPPAs construction by taking as limit points not $\varepsilon = 0$ and $\varepsilon = \infty$, but some small and large (but finite) values (Terapos and Diamessis, 1984). On the other hand, Martin and Baker (1991) (see also Chalbaud and Martin, 1992) proposed the so called quasifractional approximants (QAs). Let us suggest that we have perturbation approach in powers of ε for $\varepsilon \to 0$ and asymptotic expansions $F(\varepsilon)$ containing, for example, logarithm for $\varepsilon \to \infty$. By definition QA is a ratio R with unknown coefficients a_i, b_i, containing both powers of ε and $F(\varepsilon)$.The coefficients a, b are chosen in such a way, that (a) the expansion of R in powers of ε match the corresponding perturbation expansion; and (b) the asymptotic behavior of R for $\varepsilon \to \infty$ coincides with $F(\varepsilon)$. Let us examine the example of Laplace inverse problem. In Section 4.2 we had power series expansions for $t \to 0$ and $t \to \infty$. But this case is not general, because usually the inverse contains exponential functions. Then, we may construct QA to power expansion for $t \to 0$ and to exponential terms given for $t \to \infty$ (Andrianov, 1992). The interesting heuristic methods for multiplicative and additive matching of limiting asymptotic expansions were proposed by Chicovani et al. (1990), Frost and Harper (1976), Kalitvyasky et al. (1985), Kashin et al. (1983, 1984). Succesfull application of the quasifractional approximants to the mechanics of inhomogeneous media, for evaluation of the effective moduli, has been discussed recently by Andrianov et al. (1996).

6. Concluding remarks. Perspectives and open problems

The main advantage of the TPPAs and QAs methods is simplicity of algorithms allowing for solving the complicated problems even with using a finite number of coefficients of the asymptotic expansions. For the control of the correctness of the matching realized by the TPPAs and QAs the numerical methods (Kaas and Petersen, 1987; Krasnosel'ski et al., 1969) or procedures of recalculations of the matching parameters (Bensaadi and Potier-Ferry, 1993; Cochelin et al., 1993) may be applied. To this end one-point PAs constructed for the expansions $\varepsilon \to 0$ and $\varepsilon \to \infty$ can be used (Mason, 1964, 1981). It has been observed that the PAs possesses the property of self-correction of the errors (Luke, 1980a,b; Litvinov, 1993) and then may be used for the solution of the so called "ill-posed" problems. The PAs self-correction effect is closely connected with the fact that errors of the coefficients of PA don't spread arbitrary, but wrong coefficients create the new good approximations of the solution. Until now, we don't know, does this self-correction property exist for the TPPAs. It is worth adding that PAs, TPPAs and QAs applications bear new ideas for mathematicians, working on multi-dimensional Padé approximants (Baker and Graves-Moris, 1996). On the other hand, some well developed branches of PAs, for example branching continued fractions (Scorobogat'ko, 1987), are still waiting for applications. Evidently, as we said above, PAs, TPPAs and QAs are not a panacea, and in some cases they fail. Then, it is possible to apply other methods of interpolation (Bakhvalov et al., 1987; Krylov, 1989; Scorobogat'ko, 1987).

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Dwu-punktowe aproksymanty Padé i ich zastosowanie do rozwiązywania problemów mechaniki

Streszczenie

Wiele matematycznych i mechanicznych zagadnień zależnych od bezwymiarowego parametru ε daje się rozwiązać w postaci rozwinięć asymptotycznych wokól $\varepsilon = 0$ i $\varepsilon = \infty$. Powstaje naturalne pytanie, czy można w oparciu o dostępne rozwinięcia asymptotyczne sluszne dla $\varepsilon \to 0$ and $\varepsilon \to \infty$ przewidzieć rozwiązanie obowiązujące dla dowolnego ε. Na tak ogólnie postawione pytanie nie uzyskano dotychczas pelnej odpowiedzi. Istnieją jednak szczególne przypadki, gdzie taka odpowiedź jest możliwa. Uzyskuje się ją konstruując dla szeregów asymptotycznych tak zwane przybliżenia Padé. Niniejsza praca poświęcona jest przeglądowi wybranych problemów teorii płyt, powłok, liniowych i nieliniowych drgań, przewodnictwa cieplnego dla ośrodków niejednorodnych i innych, które w ostatnich kilku latach rozwiązano stosując przybliżenia Padé.

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