Normal Vibrations of a General Class of Conservative Oscillators

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Abstract. This paper considers normal vibrations with curvilinear trajectories in a configuration space of systems which are close to systems permitting rectilinear normal modes of vibration. Analysis of trajectories of normal vibrations in the configuration space is used.

1. Introduction

Lyapunov [1] examined nonlinear finite dimensional systems of the form

$$\dot{y} = Ay + N(y), \quad y \in R^n, \quad N : R^n \to R^n, \quad N \in C^r, \quad r \geq 1, \tag{1}$$

with an analytical first integral (where A is an $(n \times n)$ constant matrix); he assumed that the linearized system, $\dot{y} = AY$, possesses periodic solutions with natural frequencies which are not integrally related. Lyapunov proved that (1) possesses a one-parameter family of periodic solutions. To arrive at a solution, he used two approaches. One involved a power series of the amplitude of a variable, the series having coefficients periodic in time; the other was based on phase trajectories of periodic solutions. What is shown here is the possibility of generating the same tier of periodic solutions of Lyapunov systems by examining their trajectories in the configuration space. These solutions possess all the properties of normal vibrations.

Nonlinear normal vibration modes (NNMs) are a generalization of the normal (principal) vibrations of linear systems. In a normal mode a finite-dimensional system behaves like a conservative one having a single degree of freedom. In this case all position coordinates can be parametrized by any one of them as follows:

$$x_i = p_i(x) \qquad (x \equiv x_1, i = 2, 3, \dots, n), \tag{2}$$

$p_i(x)$ being analytical functions.

Kauderer [2] became a forerunner in developing quantitative methods for analyzing NNMs. The first formulation and development of the theory of NNMs can be attributed to Rosenberg and his co-workers [3–5]. Rosenberg considered n degrees of freedom conservative oscillators and defined NNMs as 'vibrations in unison', i.e., synchronous periodic motions during which all coordinates of the system vibrate equiperiodically, reaching their maximum and minimum values at the same instant of time.

Rosenberg was the first to introduce a broad class of essentially nonlinear conservative systems allowing normal vibrations with rectilinear trajectories in a configuration space of the form:

$$x_i = k_i x_i \qquad (i = 2, 3, \dots, n). \tag{3}$$

For example, 'homogeneous systems' whose potential is an even homogeneous function of the variables belong to such a class. It is interesting to note that the number of modes of normal vibrations in the nonlinear case can exceed the number of degrees of freedom of the system. This remarkable property has no analogy in the linear (non-degenerate) case. In general, one expects the trajectories of normal vibrations of nonlinear systems to be curved instead of straight lines.

For some particular cases curvilinear trajectories were defined by Rosenberg and Kuo [6] and by Rand [8]. In the paper by Manevich and Mikhlin [8] the power series method was proposed for the construction of above mentioned trajectories.

Different results and new ideas concerning NNMs of conservative systems were performed by Vito [9], Mikhlin [10], Vedenova et al. [11], Vakakis [12], Vakakis and Rand [13], Shaw and Pierre [14, 15], Nayfeh and Nayfeh [16], etc.

In [9] NNMs are approximated by harmonic functions. In [10] Padé approximations are used for an analysis of the NNMs with large amplitudes. In [11] nonlinear mode localization is studied in discrete nonlinear systems with impact nonlinearities. In [12] asymptotic methodology is used; for an oscillator with weak coupling stiffness in both localized and non-localized modes are detected. In [13] the global dynamics of strongly nonlinear systems are analyzed by means of Poincaré maps. In [14, 15] the authors reformulated the concept of NNMs for a general class of nonlinear discrete oscillators without assuming the existence of an analytic first integral of motion. Their analysis is based on the computation of invariant manifolds of motion on which the NNM oscillations take place. In [16] a computationally efficient extension of the invariant manifold methodology (complex invariant manifold formulation) is proposed.

Here, a rigorous perturbation methodology for analyzing finite amplitude NNMs of broad classes discrete nonlinear systems is presented and the effect of internal resonances is considered.

2. Normal Vibrations in Lyapunov Systems

Let us now consider an n degrees of freedom conservative system of the form

$$m_i \ddot{x}_i + \Pi_{x_i} = 0 \qquad \left(\dot{x}_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} \, , \Pi_z = \frac{\partial \Pi}{\partial z} \, , i = 1, 2, \ldots, n \right), \tag{4}$$

$\Pi = \Pi(x)$ being potential energy assumed to be a positively definite function and $x = (x_1, x_2, \ldots, x_n)^T$; power series expansion for $\Pi(x)$ begins with terms having a power of at least 2. Without reducing the degree of generalization we assume that $m_i = 1$, since this can be always ensured by dilatation of coordinates.

The energy integral of the system (4) is given by

$$\frac{1}{2} \sum_{k=1}^{n} \dot{x}_k^2 + \Pi(x_1, x_2, \ldots, x_n) = h, \tag{5}$$

where h is the fixed level of the total energy. Assume that within the region of the configuration space, bounded by a closed maximum equipotential surface $\Pi = h$, the only equilibrium position is $x_i = 0$ $(i = 1, 2, \ldots, n)$.

In order to determine the trajectories of normal vibrations (2), the following relationships can be used [4, 8]:

$$2x_i'' \frac{h - \Pi}{1 + \sum_{k=2}^{n}(x_k'^2)} + x_i'(-\Pi_x) = -\Pi_{x_i} \qquad (i = 2, 3, \ldots, n; X \equiv X_1), \qquad (6)$$

where primes represent differentiation with respect to x.

These are obtained either as Euler equations for the variational principle in the Jacobi form [17] or by elimination of time from the equations of motion (4) taking into account the energy integral (5). Equations (6) are nonlinear and non-autonomous, and have removable singular points: they are not simpler in the linear case. Nevertheless, these relationships are suitable for the determination of nearly rectilinear trajectories of normal vibrations.

An analytical extension of the trajectories up to a maximum isoenergy surface $\Pi = h$ is possible if the following conditions of orthogonality of a trajectory at $\Pi = h$ are satisfied [4]:

$$x_i \left[-\Pi_x(X, x_2(X), \ldots, x_n(X)) \right] = -\Pi_{x_i}(X, x_2(X), \ldots, x_n(X)), \qquad (7)$$

$(X, x_2(X), \ldots, x_n(X))$ being the end points of the trajectory on $\Pi = h$, where all velocities are equal to zero. If a trajectory $x_i(x)$ is defined, the law of motion with respect to time can be found using the relation:

$$\ddot{x} + \Pi_x(x, x_2(x), \ldots, x_n(x)) = 0.$$

This can be rewritten in the form

$$\ddot{x} + V'(x) = 0.$$

The function $x(t)$ now is the inversion of the quadrature

$$t + \phi = \frac{1}{\sqrt{2}} \int_X^x \frac{\mathrm{d}\xi}{\sqrt{h - V(\xi)}} \, .$$

Without loss of generality, the phase here is so selected that the initial velocity equals zero. The amplitude and the energy are related by $h = V(x)$. This equation is solvable with respect to the amplitudes X at a given energy h, provided the conditions of the closeness of all equipotential surfaces (at various values of h) holds and no equilibrium positions exist other than $x_i = 0$. This condition also assures that the function $x(t)$ is periodic [18].

From the above discussion it is concluded that normal vibrations constitute a one-parameter (in the energy h) family of periodic solutions with smooth trajectories in a configuration space. A second arbitrary parameter, i.e. the phase ϕ, may also be considered, but this requires that the condition $\dot{x}(0) = 0$.

Let us select terms of the potential energy that are quadratic in x_j and define a generating linear system $\Pi = \Pi_2 + \tilde{N}$, where the expansion of $\tilde{N}$ in terms of x_j begins with at least a power of 3. The characteristic equation of the generating linear system has but purely imaginary roots of the form $\pm i\omega_j$ $(j = 1, 2, \ldots, n)$, $i^2 = -1$. A normal form of vibrations corresponds to a pair of roots. Let us select one of these vibration forms by fixing the pair $\pm i\omega$ $(\omega = \omega_1)$.

A basic assumption made by Lyapunov in his construction of NNMs was that there are no linearized normal modes with natural frequencies which are integrally related, i.e., which satisfy relations of the form $\omega_i = p\omega_k$, $p = 1, 2, 3, \ldots$

Transforming the quadric Π_2 to a sum of squares (this transformation is equivalent to the introduction of normal coordinates in a linear system described previously), one obtains:

$$\Pi = \frac{1}{2}\,\omega^2 x^2 + \frac{1}{2}\sum_{j=2}^{n}\omega_j^2 x_j^2 + N(x, x_2, \ldots, x_n) \qquad (x \equiv x_1). \tag{8}$$

It is well known that such a transformation does not change the form of the kinetic energy $K = (1/2)\sum_{j=1}^{n} x_j^2$. A generating of normal forms of vibrations in a linear system is now determined by equations $x_j = 0$ ($j = 2, 3, \ldots, n$), $x = X\cos(\omega t + \varphi)$. By selecting the time origin so that $\dot{x} = (0)$, one obtains $\varphi = 0$. Following Lyapunov, let us assume that all variables x_j ($j = 2, 3, \ldots, n$) are of the order of magnitude c, where $c = x(0)$, $\dot{x}(0) = 0$. On having introduced c into the potential energy (8) by substituting $x_j \to cx_j$ it is seen that every homogeneous constituent of Π contains c to a power equal to the degree of homogeneity as a factor. Thus we can write:

$$\Pi = \frac{c^2}{2}\left(\omega^2 x^2 + \sum_{j=2}^{n}\omega_j^2 x_j^2\right) + \sum_{k=3}^{\infty} c^k N^{(k)}(x, x_2, \ldots, x_n), \tag{9}$$

where the functions $N^{(k)}$ represent the $O(\|x^k\|)$ terms of the potential energy. In the new coordinates we have $x(0) = X = 1$. Later the symbol (k) will be indicated the same.

Let us now use equations (6) to find the normal oscillation trajectory of a nonlinear system which, on its linear limit, becomes rectilinear, $x_j = 0$ ($j = 2, 3, \ldots, n$). The solutions of the equations will be presented as

$$x_j = \sum_{\ell=1}^{\infty} c^\ell x_{j\ell}. \tag{10}$$

Since we are dealing with trajectories in a configurational space, the variables $x_{j\ell}$ should be regarded as functions of x, rather than of time t.

Before proceeding to construct the solution it is worth noting that the singular points of equations (6) are roots of the equation:

$$h = \Pi(x, x_2, \ldots, x_n). \tag{11}$$

As the order of approximation varies so does the estimate of the energy h, since the solution $x_i(x)$, $i = 2, \ldots, n$, becomes more accurate through the inclusion of higher order terms. Suppose that the energy h_0 of the generating linear system does not coincide with the total energy h of the nonlinear system. Let us assume

$$h = c^2 h_0 + c^3 h_i \tag{12}$$

(h_0 has an order of c^2 while nonlinear terms are of greater order than c). In particular, having estimated the functions $x_i(x)$, $i = 2, \ldots, n$, up to $O(\varepsilon^p)$, for a given amplitude of vibration $x = X$, the relation (11) leads to:

$$h = \Pi\left(x, \sum_{\ell=1}^{p} c^\ell x_{1\ell}, \ldots, \sum_{\ell=1}^{p} c^\ell x_{n\ell}\right).$$

This relationship enables the finding of the energy $h^{(\ell)}$; since the energy of the linear system h_0 is known, one obtains the estimate for the energy, $h_1^{(1)}$ to be used in the next approximation.

Substituting the series (10) into (6) and collecting the lowest order terms in c, one obtains the following equations:

$$2x_{i1}'' \left(h_0 - \frac{1}{2}\,\omega^2 x^2 \right) - x_{i1}' \omega^2 x + \omega_i^2 x_{i1} + N_{x_i}^{(3)}(x, 0, \ldots, 0) = 0, \tag{13}$$

where

$$N_{x_i}^{(3)}(x, 0, \ldots, 0) \equiv \alpha_i x^2 \qquad (i = 2, 3, \ldots, n).$$

At this point the boundary conditions (7) are used. These determine the orthogonality of the trajectory to the maximum equipotential surface. The amplitude values of $x = X$ related to the energy h_0 by

$$h_0 = \frac{1}{2}\,\omega^2 X^2 \left(X_{1,2} = \pm\sqrt{\frac{2h_0}{\omega^2}} \right).$$

Retaining the lowest order terms in c, one derives the following set of $(n - i)O(c^2)$ boundary conditions:

$$-x_{i\ell}'(X)\omega^2)X + \omega_i^2 x_{i\ell}(X) + N_{x_i}^{(3)}(X, 0, \ldots, 0) = 0 \qquad (i = 2, 3, \ldots, n). \tag{14}$$

The homogeneous equations corresponding to (13) are hypergeometric equations (with two regular singular points) whose solutions have been thoroughly studied in the literature. A general solution of (13) exists in closed form. Alternatively, the analytical solution of (13) can be represented in terms of the Taylor series about the origin of the configuration space, which can then be analytically continued up to the maximum equipotential surface by satisfying boundary conditions (14). To perform this computation the approximations $x_{i1}(x)$ are expressed as

$$x_{i1} = \sum_{j=0}^{\infty} a_{ij}^{(1)} x^j. \tag{15}$$

Substituting the series into (13), the following binomial recurrent relationships for the coefficients $a_{ik}^{(1)}$ result:

$$4ha_{i2}^{(1)} + \omega_i^2 a_{i0}^{(1)} = 0,$$

$$12ha_{i3}^{(1)} - \omega_i^2 a_{i1}^{(1)} + \omega_i^2 a_{i1}^{(1)} = 0,$$

$$24ha_{i4}^{(1)} - 2\omega^2 a_{i2}^{(1)} - 2a_{i2}^{(1)}\omega^2 + \omega_i^2 a_{i2}^{(1)} + \alpha_i = 0,$$

$$40ha_{i5}^{(1)} - 6\omega^2 a_{i3}^{(1)} - 3a_{i3}^{(1)}\omega^2 + \omega_i^2 a_{i3}^{(1)} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$2(j+2)(j+1)ha_{i,j+2}^{(1)} - j(j-1)\omega^2 a_{ij}^{(1)} + j\omega^2 a_{ij}^{(1)} + \omega^2 a_{ij}^{(1)} = 0$$

$$\ldots \qquad\qquad \ldots \qquad\qquad \ldots \tag{16}$$

($h = h_0$ in this approximation).

The radius of convergence of (15) is found by examining the infinite recurrent set of equations (16),

$$R = \lim_{j \to \infty} \left| \frac{a_{ij}^{(1)}}{a_{i,j+2}^{(1)}} \right| = \lim_{j \to \infty} \frac{2(j+2)(j+1)h}{j(j-i)\omega^2} = \frac{2h}{\omega^2} \, .$$

Note here that, as shown in [5], any solution of $x_j(x)$ converging in a domain bounded by a surface $\Pi(x, x_2, \ldots, x_n) = h$, may be analytically continued up to this surface, provided that the orthogonality conditions (7) hold, which in the first approximation are determined by (14).

The recursive formulas (16) provide non-unique solutions for the coefficients of the series (15). Uniqueness of the solution is obtained by imposing the $(n-1)$ boundary conditions (14), i.e., by continuing the Taylor series expansions up to the maximum equipotential surface. Using (16), one expresses the arbitrary coefficient $a_{ij}^{(1)}$ in terms of coefficients $a_{i0}^{(1)}$ and $a_{i1}^{(1)}$. These last coefficients are computed by employing relations (14). On substituting (15) into (14) (at $x = \pm X$, where $X^2 = 2h/\omega^2$), one obtains the additional algebraic relations (at $x = \pm X$, where $X^2 = 2h/\omega^2$):

$$\mp \omega^2 x \left[\sum_{j=1}^{\infty} j a_{ij}^{(1)} (\pm X)^{j-1} \right] + \omega_i^2 \left[\sum_{j=1}^{\infty} a_{ij} (\pm X)^{j-1} \right] + \alpha_i X^2 = 0. \tag{17}$$

Introducing at this point the quantities

$$K_j^{(i)} = \frac{j^2 - \Lambda_i^2}{(j+2)(j+1)} \, , \qquad \Lambda_i = \frac{\omega_i}{\omega} \, ,$$

the solutions to the recursive relations (16) are expressed as:

$$a_{2k}^{(i)} = K_{2k-2}^{(i)} K_{2k-4}^{(i)} \ldots K_4^{(i)} \left[K_2^{(i)} K_0^{(i)} \cdot a_{i0}^{(i)} \left(\frac{\omega^2}{2h} \right)^k - \left(\frac{\omega^2}{2h} \right)^{k-1} \frac{\alpha_i}{12\omega^2} \right] ,$$

$$a_{2k+1}^{(i)} = K_{2k-1}^{(i)} K_{2k-3}^{(i)} \ldots K_3^{(i)} K_0^{(i)} \cdot a_{i0}^{(i)} \left(\frac{\omega^2}{2h} \right)^k . \tag{18}$$

Expressions (16) relate an arbitrary coefficient $a_{ij}^{(i)}$ to the leading coefficients $a_{i0}^{(i)}$ and $a_{i1}^{(i)}$. These last coefficients are determined by substituting (18) into the boundary conditions (17), resulting in the set of n non-homogeneous algebraic equations of the following form:

$$R_0^{(i)} a_{i0}^{(i)} \pm R_1^{(i)} R_{i1}^{(1)} = R_2^{(i)} \qquad (i = 2, 3, \ldots, n), \tag{19}$$

where the computation of coefficients $R_0^{(i)}$ and $R_1^{(i)}$ requires some algebraic manipulations:

$$R_0^{(i)} = 2K_0^{(i)} + 4K_0^{(i)} K_2^{(i)} + \cdots - \Lambda_i^2 (1 + K_0^{(i)} + K_0^{(i)} K_2^{(i)} + \cdots)$$

$$= -\Lambda_i^2 + (2 - \Lambda_i^2)(K_0^{(i)} + (4 - \Lambda_i^2) K_0^{(i)} K_2^{(i)} + \cdots$$

$$= -\Lambda_i^2 + \sum_{m=1}^{\infty} (2m - \Lambda_i^2) K_0^{(i)} K_2^{(i)} \ldots K_{2m-2}^{(i)}$$

$$= 2K_0^{(i)} + \sum_{m=1}^{\infty} [-2m(2m-1) + (2m)^2 - \Lambda_i^2] K_0^{(i)} K_2^{(i)} \ldots K_{2m-2}^{(i)}$$

$$= 2K_0^{(i)} + \sum_{m=1}^{\infty} [-2m(2m-1) + (2m+2)(2m+1)K_{2m}^{(i)}] K_0^{(i)} K_2^{(i)} \ldots K_{2m-2}^{(i)}$$

$$= K_0^{(i)} K_2^{(i)} \ldots K_{2m}^{(i)} \ldots ,$$

$$R_1^{(i)} = 1 + 3K_1^{(i)} + 5K_1^{(i)} K_3^{(i)} + \cdots - \Lambda_i^2 (1 + K_1^{(i)} + K_1^{(i)} K_3^{(i)} + \cdots)$$

$$= (1 - \Lambda_i^2) + (3 - \Lambda_i^2) K_1^{(i)} + (5 - \Lambda_i^2) K_1^{(i)} K_3^{(i)} + \cdots$$

$$= (1 - \Lambda_i^2) + \sum_{j=1}^{\infty} (2j - \Lambda_i^2) K_1^{(i)} K_3^{(i)} \ldots K_{2j-1}^{(i)}$$

$$= 6K_i^{(i)} + \sum_{j=1}^{\infty} [-2j(2j+1) + (2j+1)^2 - \Lambda_i^2] K_1^{(i)} K_3^{(i)} \ldots K_{2j-1}^{(i)}$$

$$= 6K_i^{(i)} + \sum_{j=1}^{\infty} [-2j(2j+1) + (2j+3)(2j+2) K_{2j+1}^{(i)}] K_1^{(i)} K_3^{(i)} \ldots K_{2j-1}^{(i)}$$

$$= K_1^{(i)} K_3^{(i)} \ldots K_{2j+1}^{(i)}. \tag{20}$$

In order to obtain unique and non-trivial solutions for the coefficients $a_{i0}^{(i)}$ and $a_{i1}^{(i)}$, it is necessary that the coefficients of the homogeneous parts of (19) satisfy the conditions $R_0^{(i)} \neq 0$, and $R_1^{(i)} \neq 0$, $i = 2, \ldots, n$. Examining the analytical expressions (20), it is concluded that in the critical case when $R_0^{(i)} = R_1^{(i)} = 0$, a subset of coefficients

$$K_j^{(i)} = \frac{j^2 - \Lambda_i^2}{(j+2)(j+1)}$$

vanishes, or equivalently, that the linearized natural frequencies of the system satisfy a resonance relation of the form $\omega_i = j\omega$ for some positive integers $j = 1, 2, \ldots$ These were precisely the cases which were eliminated from consideration in Lyapunov's analysis [1].

Let us now consider the higher order approximations. Suppose that in the series (9) single-valued solutions $x_{ik}(x)$ are computed, where $k \leq l - 1$, which are analytical functions over a closed domain bounded by the surface $\Pi = h$. Considering $O(c^l)$ terms in (6) one obtains

$$2x_{i\ell}'' \left(h_0 - \frac{1}{2} \omega^2 x^2 \right) + 2x_{i,\ell-1}'' h_1^{(\ell-1)} - x_{i\ell}' \omega^2 x + x_{i\ell} \omega_i^2$$

$$+ \sum_{k=1}^{\ell-1} x_{ik}'' R_{\ell-k,i}^{(1)} + \sum_{k=1}^{\ell-1} x_{ik}' R_{\ell-k,i}^{(2)} - [R_{\ell-1,i}^{(3)}] = 0, \tag{21}$$

where

$$R_{pi}^{(s)} = \sum_{r=1}^{p} \sum \delta_{(\gamma)} \frac{\partial^\gamma P_{ri}^{(s)}(x, 0, \ldots, 0)}{\partial x_2^{\alpha_{2r}} \partial x_3^{\alpha_{3r}} \ldots \partial x_n^{\alpha_{nr}}} \prod_{j=2}^{n} \prod_{m=1}^{r} (x_{jm})^{\beta_{pi}}$$

($s = 1, 2, 3$), and the second summation sign in the above expression is carried out over all positive integer solutions of the equation

$$\sum_{j=2}^{n}(\beta_{1j} + 2\beta_{2j} + \cdots + r\beta_{rj}) = r,$$

with

$$\sum_{m=1}^{n}\beta_{mj} = \alpha_{jr}, \quad \sum_{j=2}^{n}\alpha_{jr} = \gamma, \quad \delta_{(\gamma)} = \frac{r!}{\Pi_{m=1}^{r}(\alpha_{jm})!(m!)^{\alpha_{jr}}}.$$

Here

$$P_{ri}^{(1)} = -2\Pi^{(r)}(x, x_2, \ldots, x_n),$$

$$P_{ri}^{(2)} = \left[-\Pi_x(x, x_2, \ldots, x_n)\left[1 + \sum_{j=2}^{n}(x_j')^2\right]\right]^{(r)},$$

$$P_{ri}^{(3)} = \left[-\Pi_{x_i}(x, x_2, \ldots, x_n)\left[1 + \sum_{j=2}^{n}(x_j')^2\right]\right]^{(r)}.$$

Equations (21) are complemented by the following set of $O(c^1)$ boundary orthogonality conditions:

$$\left[x_{i1}''\omega^2 x + x_{i1}\omega_i^2 + \sum_{k=1}^{\ell-1}x_{ik}'R_{\ell-k,i}^{(2)} - R_{\ell-1,i}^{(3)}\right]\Bigg|_{x=X_j} = 0,$$

where $x = X_j$ ($j = 1, 2$) are vibration amplitudes (one of these values, $X_1 = 1$).

The equations (21) may be rearranged as follows:

$$2x_{i\ell}''\left(h_0 - \frac{1}{2}\omega^2 x^2\right) - x_{i\ell}'\omega^2 x + x_{i\ell}\omega_i^2 + F_i^{(1)}(x) = 0, \tag{22}$$

where the terms $F_i^{(\ell)}(x)$ consist of already-computed functions of x. Expressing the solution of (22) in the series form:

$$x_{i\ell} = \sum_{j=0}^{\infty}a_{ij}^{(\ell)}x^j,$$

one obtains, similarly to the first approximation case, a non-homogeneous recurrent set of linear equations governing the coefficients $a_{ij}^{(\ell)}$. The indefinite coefficients $a_{i0}^{(\ell)}, a_{i1}^{(\ell)}$ are obtained from the boundary conditions for this approximation (conditions of orthogonality). Next the $O(c^{\ell})$ approximation for the total energy, $h^{(\ell)}$ is determined from (11), while the equation $h^{(\ell)} = c^2 h_0 + c^3 h_1^{(\ell)}$ is used to find the value of $h_1^{(\ell)}$ which appears in the calculations of the next order of approximations.

Needless to say, an alternative series of calculations would be also acceptable, namely, determining the maximum amplitude X given a fixed level of total energy h.

The conclusions concerning the convergence and unambiguous definitions of the coefficients arrived at for the series in the first approximation hold for the series in the ℓ-th approximation as well.

Let us consider the convergence of the series (10). As shown, a series of the form (10) represents a single-valued (provided $K_j^{(i)} \neq 0$ holds) formal solution of a boundary problem (6), (7) with the coefficients analytical in x. In this case $x_j(0) = a_{j1}$, $x_j'(0) = a_{j2}$, where a_{j1}, a_{j2} can be made sufficiently small by choosing a small value of the parameter c.

Over a domain $\Pi < h$ all functions involved in (5) are analytical in x. Therefore, it follows from Poincaré's theorem on the small parameter series expansion [19] that there is a value of $c_0 > 0$ such that, for all $|c| < c_0$, the series of the form (10) converges in the domain, represents a unique solution of (6) analytical in c and x, and satisfies the conditions $x_j(0) = a_{j1}$, $x_j'(0) = a_{j2}$; moreover, as $c \to 0$ this solution becomes the trivial generating solution $x_{j0} = 0$ ($j = 2, 3, \ldots, n$). Since series of the form (10) also satisfy the conditions (7), the solution can be analytically continued up to the domain boundary $\Pi = h$.

On having obtained the smooth trajectory, the problem of finding a periodic solution reduces to the integration of a conservative system with one degree of freedom.

Concluding this treatment of normal vibrations in Lyapunov systems, we note the following.

Firstly, the trajectories $x_j(x)$ can be derived not only in terms of power series in x, but also by the method of successive approximations. This method is described below for certain cases of nonlinearity of the generating system.

Secondly, the requirement of the existence of an energy integral (5) is not essential. In autonomous systems of the form

$$\ddot{x}_i = f_i(x_1, x_2, \ldots, x_n)$$

the equations for obtaining the trajectories $x_i(x)$, ($x \equiv x_1$) take the form

$$x_i'' \dot{x}^2 + x_i' f_1(x, x_2, \ldots, x_n) = f_i(x, x_2, \ldots, x_n). \tag{23}$$

If an analytical first integral $H(x, \dot{x}, x_2, \dot{x}_2, \ldots, x_n, \dot{x}_n) = 0$ exists, all that is needed is to obtain $\dot{x}$ as a single-valued analytical function of x, x_2, x_2', $\ldots$, x_n, x_n', and to substitute the expression obtained into (23). Equations (23) can then be analyzed similarly as above.

Finally, it should be noted that an approach involving the examination of trajectories in a configurational space is by no means more complicated than approaches employing series with coefficients periodic in time or examining phase trajectories.

3. Normal Oscillations in Conservative Systems Close to Systems Admitting Rectilinear Modes

Consider now a system of a more general class, namely one with a nonlinear generating system. The smallness of disturbances will be evaluated using a small parameter ε.

The equations of motion are expressed in the form

$$\ddot{q}_i + F_{q_i}^0(q_1, q_2, \ldots, q_n) + \varepsilon F_{q_i}^1(q_1, q_2, \ldots, q_n) = 0 \qquad (i = 1, 2, \ldots, n),$$

where $F_{q_i}^0 + \varepsilon F_{q_i}^1$ is the analytical potential energy, and the unperturbed systems ($\varepsilon = 0$) possesses rectilinear modes of vibrations $q_i = k_i q$ ($q \equiv q_1$, $i = 2, 3, \ldots, n$).

Consider one of these solutions as a generating one. Rotating the coordinate axes, so that the new x-axis is directed along the rectilinear trajectory, and the remaining coordinate axes

are orthogonal to it. A generating solution in the new coordinates is represented as $x_i = 0$ $(i = 2, 3, \ldots, n)$, $x_1 \equiv x = x(t)$.

In particular, for a system with two degrees of freedom

$$\ddot{q}_i + F_{qi}(q_1, q_2) = 0 \qquad (i = 1, 2),$$

allowing the solution $q_2 = kq_1$, the formulae for the axes rotations are of the form

$$x_1 = q_1 \cos \vartheta + q_2 \sin \vartheta, \quad x_2 = -q_1 \sin \vartheta + q_2 \cos \vartheta \qquad (\vartheta = \arctan k). \tag{24}$$

The equations of motion in the new coordinates take the following form:

$$\ddot{x}_i + \Pi_{x_i}(x_1, x_2) = 0 \qquad (i = 1, 2),$$

where

$$\Pi_{x_1} = F_{q_1} \cos \vartheta + F_{q_2} \sin \vartheta, \quad \Pi_{x_2} = -F_{q_1} \sin \vartheta + F_{q_2} \cos \vartheta.$$

Let us assume that in the general case (with n degrees of freedom) after the rotation of the axes, the system

$$\ddot{x}_i + \Pi_{x_i}^0(x_1, x_2, \ldots, x_n) + \varepsilon \Pi_{x_i}^1(x_1, x_2, \ldots, x_n) = 0 \tag{25}$$

also admits a generating solution at $x_i = 0$ at $\varepsilon = 0$; but this means that

$$\Pi_{x_i}^0(x_1, 0, \ldots, 0) \equiv 0 \qquad (i = 2, 3, \ldots, n). \tag{26}$$

It is initially assumed that the unperturbed system is homogeneous, i.e., Π_0 is an even homogeneous function of the power of $r + 1$ in all the variables (r may take the following values $r = 1, 3, 5, \ldots$).

Note that, in similarity to the Lyapunov systems examined in the previous section, the small parameter could be chosen, to scale the amplitude of vibration $c = x(0)$. Substituting for $x_i \to cx_i$, one then selects a generating homogeneous system containing the smallest powers of the positional variables. This generating system may be nonlinear as well if $r > 1$.

In order to determine the trajectories of normal oscillations in the configurational space, another transition is required, this time from (25) to (6) used in combination with the boundary conditions (7).

A solution is sought in the form of a small parameter series:

$$x_i = \sum_{k=1}^{\infty} \varepsilon^k x_{ik}(x) \qquad (x \equiv x_1). \tag{27}$$

Although in this case the generating system is essentially nonlinear, all computations are similar to those performed for the linearizable case.

Let us assume that the total energy h of the entire system and the energy h_0 of the generating system are related as follows:

$$h = h_0 + \varepsilon h_1.$$

Substituting (27) into (6), one isolates the ℓ-th approximation equations in ε:

$$2x_{i\ell}'' \left[h_0 - \Pi^0(x, 0, \ldots, 0) \right] + 2x_{i,\ell-1}'' \left[h_1^{(\ell-1)} + x_{i\ell}' \Pi_x^0(x, 0, \ldots, 0) \right]$$

$$+ \sum_{k=2}^{n} \Pi_{x_i x_k}^0(x, 0, \ldots, 0)x_{k\ell} + \sum_{j=1}^{\ell-1} x_{ij}'' R_{\ell-j,i}^{(1)} + \sum_{k=1}^{\ell-1} x_{ik}' R_{\ell-k,i}^{(2)} + R_{\ell-1,i}^{(3)} = 0, \tag{28}$$

where

$$R_{mi}^{(s)} = \sum \delta_{(\gamma)} \frac{\partial^\gamma P_{ri}^{(s)}(x, 0, \ldots, 0)}{\partial x_2^{\alpha_{2m}} \partial x_3^{\alpha_{3m}} \ldots \partial x_n^{\alpha_{nm}}} \prod_{j=2}^{n} \prod_{p=1}^{m} (x_{jp})^{\beta_{pj}} \qquad (s = 1, 2, 3).$$

The $\sum$ sign in the above expression is carried over all positive integer solutions of the equation where

$$\sum_{j=1}^{n-1} (\beta_{1j} + 2\beta_{2j} + \cdots + m\beta_{mj}) = m$$

with

$$\sum_{p=1}^{m} \beta_{pj} = \alpha_{jm}, \qquad \sum_{j=1}^{n-1} \alpha_{jm} = \gamma, \quad \delta_{(\gamma)} = \frac{m!}{\Pi_{p=1}^{m}(\alpha_{jr})!(p!)^{\alpha_{jr}}} \ .$$

Here

$$P_{ri}^{(1)} = -2\Pi^{(r)}(x, x_2, \ldots, x_n),$$

$$p_{ri}^{(2)} = \left[-\Pi_x(x, x_2, \ldots, x_n) \left[1 + \sum_{k=2}^{n} (x'_k)^2 \right] \right]^{(r)},$$

$$P_{ri}^{(3)} = \left[-\Pi_{x_i}(x, x_2, \ldots, x_n) \left[1 + \sum_{k=2}^{n} (x'_k)^2 \right] \right]^{(r)}.$$

The boundary conditions (conditions of orthogonality) corresponding to this approximation assume the following form:

$$\left[-x'_{i\ell}\Pi_x^0(x, 0, \ldots, 0) - \sum_{k=2}^{n} \Pi_{x_i x_k}^0(x, 0, \ldots, 0)x_{k\ell} \right.$$

$$\left. + \sum_{k=1}^{\ell-1} x'_{ik} R_{\ell-k,i}^{(2)} - R_{\ell-1,i}^{(3)} \right]\bigg|_{x=X_j} = 0 \qquad (j = 1, 2), \tag{29}$$

with the amplitudes X_j ($j = 1, 2$) and the energy h being computed by $h = \Pi(x, x_2, \ldots, x_n)$ where x_k ($k = 2, 3, \ldots, n$) should be expressed by the series (27) worked out to an order of $\varepsilon^{\ell-1}$.

Since the unperturbed system is homogeneous, the matrix $B = \Pi_{x_i x_k}^0(x, 0, \ldots, 0)$ may be written as $B = b_{ik}x^{r-1}$. Note that, owing to the conservative nature of the system, $b_{ik} = b_{ki}$, and a symmetric matrix b_{ik} is reduced to a diagonal form by an invertible linear transformation of coordinates [20]. Therefore, without loss of generality, one can assume that in (28) and (29) the function $\Pi_{x_i x_k}^0(x, 0, \ldots, 0) = 0$ for $i \neq k$. Hence, only the terms $\Pi_{x_i x_i}^0(x, 0, \ldots, 0)x_{i\ell}$ are retained in the expression $\sum_{k=2}^{n} \Pi_{x_i x_k}^0(x, 0, \ldots, 0)x_{k\ell}$, and the set (28), together with the boundary conditions (29) is 'split' in the variables $x_{i\ell}$.

Having substituted the series $x_{i\ell} = \sum_{j=0}^{\infty} a_{ij}^{(\ell)}x'$ into (28), one finds that the coefficients $a_{ij}^{(\ell)}$ are interrelated by an infinite set of recurrent relationships

$$2h_0(r + j + 2)(r + j + 1)a_{i,r+j+2}^{(\ell)} - j(j + 1)2\Pi^0(1, 0, \ldots, 0)a_{i,j+1}^{(\ell)}$$

$$- (j + 1)\Pi_x^0(1, 0, \ldots, 0)a_{i,j+1}^{(\ell)} + \Pi_{x_i x_i}^0(1, 0, \ldots 0)a_{i,j+1}^{(\ell)} = \phi_{ij}^{(\ell)}, \tag{30}$$

where $\phi_{ij}^{(\ell)}$ denotes the terms that depend on the solutions of preceding approximations. The recurrent relationships can be used to express all coefficients $a_{ij}^{(\ell)}$ in terms of the leading coefficients $a_{i0}^{(\ell)}$ and $a_{i1}^{(\ell)}$ ($i = 2, 3, \ldots, n$). The convergence of the series is ascertained precisely as it is done with Lyapunov systems over a domain $\Pi(x, x_2, \ldots, x_n) < h$. An analytical continuation of the solution up to the 'boundary' $\Pi(x, x_2, \ldots, x_n) = h$ may be effected, provided that the boundary conditions are satisfied. Substituting the series $x_{i\ell}(x)$ into these boundary conditions (29), and in view of the recurrent relationships (30), one obtains the equations governing $a_{i0}^{(\ell)}$, $a_{i1}^{(\ell)}$.

$$\sum_{i=2}^{n} (R_{0i} a_{i0}^{(\ell)} \pm R_{1i}^{(\ell)} a_{i1}^{(\ell)}) = R_{2i}^{(1)} \qquad (i = 2, 3, \ldots, n).$$

As in the case of Lyapunov systems, the determinants of coefficients may be represented as products of an infinite number of factors

$$K_p = |q_{ij}|,$$
$$q_{ij} = \delta_i^j \left[p(p-1)2\Pi^0(1, 0, \ldots, 0) + p\Pi_x^0(1, 0, \ldots, 0) - \Pi_{x_i x_i}^0(1, 0, \ldots, 0) \right], \qquad (31)$$

where δ_i^j are the Kronecker's delta, $p = 0, 1, 2, 3, \ldots$

When the generating system is linear, the solvability conditions (31) can be shown to degenerate to the conditions of absence of internal resonances. Hence, the conditions $K_p \neq 0$ can be viewed as generalizations of the conditions of the absence of internal resonances derived in the linearizable case, and ensure that the analytical, asymptotic solutions $x_j = x_j(x)$ are unique and single-valued.

The series $x_j^{(\ell)} = \sum_{k=1}^{\ell} \varepsilon^k x(x_{jk})$ ($j = 2, 3, \ldots, n$) may now be substituted into the equation $h = \Pi(x, x_2, \ldots, x_n)|_{x=X}$ and a refined value of the total energy approximation $h^{(\ell)}$ may be obtained at a given amplitude $x = X$ ($\dot{x} = 0$). Hence, $h_1^{(\ell)}$ may be deduced from $h = h_0 + \varepsilon h_1$ to be employed in the next approximation in ε.

The considerations concerning the convergence of the series (27) given in the preceding section still hold.

Note that in the absence of the assumption of 'splitting' of the first approximation equations, the conditions (31) for the generating of homogeneous systems become different. In the general case they take the following form:

$$K_p = |q_{ij}| \neq 0,$$
$$q_{ij} = \delta_i^j \left[p(p-1)2\Pi^0(1, 0, \ldots, 0) + p\Pi_x^0(1, 0, \ldots, 0) - \Pi_{x_i x_j}^0(1, 0, \ldots, 0) \right],$$
$$(p = 0, 1, 2, 3, \ldots). \qquad (32)$$

It is worth noting that at $r = 1$ the conditions (32) become $\omega_i \neq p\omega$ which for Lyapunov systems meant that in this case inner resonances in the generating linear system are not considered.

Proceeding to non-homogeneous generating systems with the potential energy Π^0, we shall confine the discussion with the first approximation in ε:

$$2x_{i\ell}'' \left[h - \Pi^0(x, 0, \ldots, 0) \right] + x_{i\ell}'' \left[-\Pi_x^0(x, 0, \ldots, x) \right]$$

$$+ \sum_{k=2}^{n} \Pi^0_{x_i x_k}(x, 0, \ldots, 0) x_{k\ell} + \Pi^1_{x_i}(x, 0, \ldots, 0) = 0 \qquad (i = 2, 3, \ldots, n). \tag{33}$$

The homogeneous part of the set (33) contains variational equations for the generating of normal vibrations. Let us assume that the variational equation set may be 'split' into n independent equations by an invertible linear transformation with constant coefficients. The possibility of such 'splitting' is discussed in more detail in [21, 22].

Here it will be shown that, for the case of two degrees of freedom, it is always possible to effect the 'splitting' by rotating the axes (24). Indeed, introduce the transformation (24), and expand the potential energy $\Pi^0(x_1, x_2)$ into an x_2 ($x_1 \equiv x$) power series:

$$\Pi^0(x, x_2) = \Pi^0(x, 0) + \Pi^0_{x_2}(x, 0)x_2 + \frac{1}{2!}\Pi^0_{x_2 x_2}(x, 0)x_2^2 + \frac{1}{3!}\Pi^0_{x_2 x_2 x_2}(x, 0)x_2^3 + \cdots$$

The condition of existence of a normal vibration form $x_2 = 0$ is given by $\Pi^0_{x_2}(x, 0) = 0$, and the equations of motion take the form

$$\ddot{x} + \Pi^0_x(x, 0) + \frac{\mathrm{d}}{\mathrm{d}x}\Pi^0_{x_2 x_2}(x, 0)\frac{x_2^2}{2} + O(x_2^3) = 0,$$

$$\ddot{x} + \Pi^0_{x_2 x_2}(x, 0)x_2 + \frac{\mathrm{d}}{\mathrm{d}x}\Pi^0_{x_2 x_2 x_2}(x, 0)\frac{x_2^2}{2} + O(x_2^3) = 0.$$

Here, the variational equations in u and v for the normal vibrations $x = x(t)$, $x_2 = 0$ are as follows:

$$\ddot{u} + \frac{\mathrm{d}}{\mathrm{d}x}\Pi^0_{x_2 x_2}(x, 0)u = 0,$$

$$\ddot{v} + \Pi^0_{x_2 x_2}(x, 0)v = 0. \tag{34}$$

The set of equations is thus 'split'.

Introducing a new independent variable x instead of t, one obtains, in the place of (34),

$$2v'' \left[h - \Pi^0_x(x, 0)\right] + v' \left[-\Pi^0_x(x, 0)\right] + \Pi^0_{x_2 x_2}(x, 0)v = 0. \tag{35}$$

This equation corresponds to (33) at $n = 2$ when the latter retains only a homogeneous constituent of the potential energy.

Since a rectilinear normal mode of vibration has only two cusps, the kinetic energy $k = h - \Pi^0(x, 0)$ vanishes twice over one period. Therefore, equation (35) has two regular finite singular points on the real axis.

There exist classes of potentials for which equation (35) has been studied so thoroughly that it is possible to develop a system of solutions either in power series (with due regard to singularities), in trigonometric series, or in the form of series in certain special functions. If $\Pi^0(x_1, x_2)$ is a homogeneous even function, (35) may even be reduced to a hypergeometric equation by the substitution $x^p = z$, where p is the degree of homogeneity [22]. If the potential energy $\Pi^0(x, x_2)$ contains terms of second and fourth powers of x and x_2, (35) is the Lame equation [23]. Denote by $\{w_1(x), w_2(x)\}$ a fundamental set of solutions for (35), and it becomes possible to construct a general solution of the first approximation equation in ε (33):

$$x_{21} = C_1^{(1)}w_1(x) + C_2^{(1)}w_2(x) + w_1(x)\int_X^x \frac{\Pi^1_{x_2}(x, 0)w_2}{\Delta}\,\mathrm{d}x - w_2(x)\int_X^x \frac{\Pi^1_{x_2}(x, 0)w_1}{\Delta}\,\mathrm{d}x,$$

where

$$\Delta = \begin{vmatrix} w_1 & w_1 \\ w'_1 & w'_2 \end{vmatrix} = \text{const.},$$

and w_1, w_2 are either hypergeometric or Lame functions.

The constants $C_1^{(1)}$, $C_2^{(1)}$ should be determined from the regularity conditions for the solution in any point of the interval $\Pi^0(x, 0) \leq h$. Suppose that the powers of the singular points X_j ($j = 1, 2$) (zero kinetic energies) be equal to $(0, \alpha_1)$ and $(0, \alpha_2)$, respectively. In particular, for the Lame equation $\alpha_1 = \alpha_2 = 1/2$. Then, in the vicinity of these points the general solution is decomposed as follows:

$$x_{21} = A_1(C_1, C_2)f_1(x) + A_2(C_1, C_2)f_2(x)(x - X_1)^{\alpha_1},$$

$$x_{21} = B_1(C_1, C_2)g_1(x) + B_2(C_1, C_2)g_2(x)(x - X_2)^{\alpha_2}.$$

Evidently, the regularity conditions for this solution are of the form

$$A_2(C_1, C_2) = B_2(C_1, C_2) = 0.$$

A solution analytical in x is similarly derived in the next approximation in ε obtained either through the construction of vibration normal modes as ε series or by the iteration method.

A generalization to the case of n degrees of freedom ($n > 2$) is evident, provided the set of variational equations can be 'split'.

Some applications of the asymptotic methodology for approximating the NNMs are presented in [12, 24]. In particular, in [12] the asymptotic methodology is implemented to analyze the free vibrations of two unit masses connected by means of three strongly nonlinear stiffnesses with cubic nonlinearity. The zeroth order and $O(\varepsilon)$ approximations to NNMs are calculated.

The systems considered above can be obtained in calculations of nonlinear vibrations of rods, strings, plates, shells and other elastic systems (using the Bubnov–Galerkin technique) (see, for instance, [2, 25]). Some examples of the NNMs analysis in nonlinear elastic systems using Bubnov–Galerkin discretization are presented in [26].

4. Conclusions

Normal modes of vibrations with curvilinear trajectories in the configuration space are considered. The corresponding boundary problem is formulated. It is shown that such normal modes may be constructed for the Lyapunov systems and for a more general class of conservative systems, neighboring systems possessing NNMs with rectilinear trajectories. The convergence of the power series obtained was considered. Under some conditions (that generalize the conditions of internal resonances in the quasilinear case) single-valued solutions do not exist.

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