



Normal Vibrations in Near-Conservative Self-Excited and Viscoelastic Nonlinear Systems

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Abstract. A perturbation methodology and power series are utilized to the analysis of nonlinear normal vibration modes in broad classes of finite-dimensional self-excited nonlinear systems close to conservative systems taking into account similar nonlinear normal modes. The analytical construction is presented for some concrete systems. Namely, two linearly connected Van der Pol oscillators with nonlinear elastic characteristics and a simplest two-degrees-of-freedom nonlinear model of plate vibrations in a gas flow are considered.

Periodical quasilinear solutions of integro-differential equations corresponding to viscoelastic mechanical systems are constructed using a convergent iteration process. One assumes that conservative systems appropriate for the dominant elastic interactions admit similar nonlinear normal modes.

Keywords: Self-excited nonlinear systems, nonlinear normal modes (NNMs), viscoelastic nonlinear systems, power series, iterations.

1. Introduction

Lyapunov proved that nonlinear finite-dimensional systems with an analytical first integral allow a one-parameter family of periodic solutions which tend towards linear normal vibration modes as amplitudes tend to zero. Natural frequencies of corresponding linearized systems must not be integrally related [1]. The Lyapunov's solutions possess all the properties of linear normal modes [2].

Nonlinear normal vibrations modes (NNMs) are a generalization of the normal (principal) vibrations of linear systems. In the normal mode, a finite-dimensional system behaves like a conservative one in having a single degree of freedom, and all position coordinates can be analytically parametrized by any one of them.

Rosenberg, in a series of papers [3–5] provided analytical methods for computing NNMs. He defined them as 'vibrations in unison' and introduced a broad class of essentially nonlinear conservative systems allowing for NNMs with rectilinear trajectories in a configuration space ('similar' NNMs).

For example, 'homogeneous systems' whose potential is an even homogeneous function of the variables belong to such a class. It is interesting to note that the number of the NNMs may exceed the number of degrees of freedom of the oscillator. In general, the NNMs trajectories are curvilinear instead of straight lines in linear systems.

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For some particular cases, curvilinear trajectories were analyzed by Rosenberg and Kuo [5] and by Rand [6]. The power series method was proposed by Manevich and Mikhlin [7] for the construction of the curvilinear trajectories of NNMs.

Later, new results concerning NNMs of conservative systems were performed by several authors, including Vedenova et al. [8], Vakakis [9], Vakakis and Rand [10], Shaw and Pierre [11, 12], Nayfeh and Nayfeh [13], and Mikhlin [14].

In [8], nonlinear normal mode localization is studied in non linear systems with impact nonlinearities. In [9], for an oscillator with weak coupling stiffness, both localized and non-localized modes are detected using an asymptotic methodology. In [10], the NNMs and global dynamics of nonlinear systems are analyzed by means of Poincaré maps. In [11, 12], the authors reformulated the concept of NNMs for a general class of nonlinear discrete oscillators. The analysis is based on the computation of invariant manifolds of motion on which the NNM oscillations take place. In [13], a complex invariant manifold formulation is proposed. In [14], Padé approximations are used for an analysis of NNMs with large amplitudes.

Note that publications on the generalization of NNMs to nonconservative systems are not numerous.

It is well known that forced vibrations in nonlinear systems with one degree of freedom and small periodic disturbances are similar to those in an undisturbed conservative system (in the resonance region).

It should be noted that periodic solutions in nonautonomous systems close to Lyapunov systems were thoroughly investigated by Malkin [15].

The Rauscher method was used for analyses of normal vibrations in nonautonomous systems by Kinney and Rosenberg [16]. To solve the problem, one assumes that the external periodic force is proportional to some chosen positional coordinate raised to such a power that the resultant autonomous system is homogeneous.

NNMs in general finite-dimensional nonautonomous systems close to conservative systems allowing similar NNMs were considered by Mikhlin [17]. Rauscher's ideas and the power-series method for trajectories in a configurational space are used in the construction of resonance solutions.

Some other works on NNMs in nonautonomous systems, have been published by Yang and Rosenberg [18], Szemplinska-Stupnicka [19], and Vakakis and Caughey [20].

Of particular note is the work by Rand and Holmes [21], where periodic solutions of two weakly coupled Van der Pol oscillators are analyzed. Note that NNMs in a model of two strongly coupled Van der Pol oscillators are considered in this work.

Basic and new results on NNMs are presented in the book by Vakakis et al. [22] which describes quantitative and qualitative analyses of NNMs in conservative and nonautonomous systems, including localized modes, an analysis of stability, a generalization, and an analysis of NNMs in distributed systems.

Here, the perturbation methodology is utilized for the analysis of NNMs in broad classes of finite-dimensional self-excited nonlinear systems close to conservative systems allowing normal vibrations with rectilinear trajectories in a configuration space (Section 1). The analytical construction of NNMs is presented in Section 2 for some concrete systems. The first of them represents two linearly connected Van der Pol oscillators with nonlinear elastic characteristics. Another example refers to the problem of plate vibrations in the flow of gas. A simple, two-degrees-of-freedom nonlinear model is considered. In Section 3, periodical quasnormal solutions of integro-differential equations corresponding to viscoelastic mechanical systems are constructed. One assumes that elastic forces are dominant and conservative

systems appropriate for the elastic interactions allow similar NNMs which are selected as generative solutions. A convergent iteration process is used here.

2. Normal Vibrations in Near-Conservative, Self-Excited Nonlinear Systems

2.1. EQUATIONS DESCRIBING THE TRAJECTORIES OF NNMS

We will demonstrate that the approaches for analysis of NNMs in conservative systems are applicable to near-conservative autonomous systems with small self-excited perturbations. We will also consider modes when all positional coordinates of the finite-dimensional, self-excited system are linked. In relation to these modes, the system behaves like a single-degree-of-freedom one. The periodic solutions close to NNMs in the generating conservative system could be called *nonlinear normal modes (NNMs) of the self-excited nonlinear system*.

Consider the following near-conservative system:

$$\ddot{x}_i + \Pi_{x_i}(x_1, x_2, \dots, x_n) + \varepsilon g_i(x_1, \dot{x}_1, \dots, x_n, \dot{x}_n) = 0, \quad (2.1)$$

where ε is a small parameter and the functions g_i may be nonlinear with respect to \dot{x}_i . The functions f_i and Π are assumed to be analytical in x_i, \dot{x}_i ; a potential energy Π is subject to some limitations which will be shown later. The system may involve friction of any physical nature, such as viscous, dry, or turbulent. The conditions that ensure the occurrence of self-excited vibrations (limiting cycles) are discussed later.

Assume that a conservative system ($c = 0$) permits similar NNMs, i.e., normal modes with rectilinear trajectories in a configuration space: $x_{i0} = k_i x_{i0}$ ($i = 2, \dots, n$; k_i are constants). Any terms may be regarded as a perturbation.

Consider one of these NNMs as a generating one. Rotating the coordinate axes so that the new x -axis is directed along the rectilinear trajectory, one obtains a generating solution in the new coordinates as

$$x_{i0} \quad (i = 2, 3, \dots, n), \quad x_{i0} \equiv x = x(t). \quad (2.2)$$

Select some positional variable $x \equiv x_i$.

Let us find a solution that all phase coordinates are defined as single-valued and analytical functions of x :

$$x_i = x_i(x, \varepsilon), \quad \dot{x} = \dot{x}(x, \varepsilon), \quad \dot{x}_i = \dot{x}_i(x, \varepsilon). \quad (2.3)$$

Introducing a new independent variable x instead of t , one obtains from (2.1) the equations describing the trajectories of NNMs:

$$\begin{aligned} x_i''(\dot{x}(x))^2 + x_i'[-\Pi_x(x, x_2(x), \dots, x_n(x)) - \varepsilon g_1(x, \dot{x}(x), x_2(x), \dot{x}_2(x), \dots)] \\ + \Pi_{x_i}(x, x_2(x), \dots, x_n(x)) + \varepsilon g_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x), \dots) = 0 \\ (i = 2, 3, \dots, n). \end{aligned} \quad (2.4)$$

Here and henceforth, a prime denotes differentiation with respect to x ; it is clear that $\dot{x}_i(x, \varepsilon) = x_i'(x, \varepsilon)\dot{x}$.

Select the trajectory return points where all velocities are equal to zero. When $x = X_j$ ($j = 1, 2$) and $\dot{x} = 0$, we obtain the additional conditions from (2.4):

$$\begin{aligned} & \left\{ x'_i [-\Pi_x(x, x_2(x), \dots, x_n(x)) - \varepsilon g_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x) \dots)] \right. \\ & \left. + \Pi_{x_i}(x, x_2(x), \dots, x_n(x)) + \varepsilon g_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x) \dots) \right\} \Big|_{x=X_j} = 0 \\ & (i = 2, 3, \dots, n; j = 1, 2). \end{aligned} \quad (2.5)$$

Here $(X, x_2(X), \dots, X_n(X))$ being the trajectory return points. Equations (2.4) in combination with conditions (2.5) allow a univalent determination of $x_i(x)$. Note that equations similar in form and additional boundary conditions were previously considered in the case of conservative systems [2–4, 22, 23].

In the zero approximation ($\varepsilon = 0$) $\dot{x}_0^2 = 2[h_0 - \Pi(x, 0, \dots, 0)]$, where h_0 is the energy of the nonperturbed conservative system. We shall restrict our consideration of (2.4) to the first approximation in ε :

$$\begin{aligned} & 2x''_{i1}[h_0 - \Pi(x, 0, \dots, 0)] + x'_{i1}[-\Pi_x(x, 0, \dots, 0)] \\ & + \sum_{k=2}^n x_{k1} \Pi_{x_i x_k}(x, 0, \dots, 0) + g_i(x, \dot{x}_0(x), 0, \dots, 0) = 0 \\ & (i = 2, 3, \dots, n). \end{aligned} \quad (2.6)$$

In this approximation, the boundary conditions are obtained from (2.4) at $\dot{x}_0 = 0$, $x = X_j$ ($j = 1, 2$):

$$\begin{aligned} & \left\{ x'_{i1}[-\Pi_x(x, 0, \dots, 0)] + \sum_{k=2}^n x_{k1} \Pi_{x_i x_k}(x, 0, \dots, 0) \right. \\ & \left. + g_i(x, \dot{x}(x), 0, \dots, 0) \right\} \Big|_{x=X_{1,2}} = 0 \\ & (i = 2, 3, \dots, n). \end{aligned} \quad (2.7)$$

The power series method may be applied to find $x_{i1}(x)$.

2.2. THE POWER SERIES METHOD

It is assumed, for example, that the generating conservative system is homogeneous, i.e. Π is an even homogeneous function of the power of $r + 1$ in all the variables (r may take the values $r = 1, 3, 5, \dots$).

Since the unperturbed system is homogeneous, the matrix $B = \Pi_{x_i x_k}(x, 0, \dots, 0)$ may be written as $B = b_{ik} X^{r-1}$. Note that owing to the conservative nature of the system, $b_{ik} = b_{ki}$, and the symmetric matrix b_{ik} is reduced to a diagonal form by nondegenerate linear transformation of the positional coordinates [24]. The transformation can be chosen in order to preserve a generating solution in the form (2.1). Therefore, without loss of generality, one can assume that in (2.6) and (2.7) the function $\Pi_{x_i x_k}(x, 0, \dots, 0) = 0$ for $i \neq k$. Hence, the set (2.6), together with the boundary conditions (2.7) is ‘split’ in the variables x_{i1} .

Having substituted the series

$$x_{i1} = \sum_{j=0}^{\infty} a_{ij} x^j \quad (2.8)$$

into (2.6), one finds that the coefficients a_{ij} are interrelated by the following infinite set of recurrent relationships (it is written out coefficients at x^{r+j}):

$$\begin{aligned} & 2h(r+j+2)(r+j+1)a_{i,r+j+2} - j(j+1)2\Pi(1, 0, \dots, 0)a_{i,j+1} \\ & - (j+1)\Pi_x(1, 0, \dots, 0)a_{i,j+1} + \Pi_{x_i x_i}(1, 0, \dots, 0)a_{i,j+1} = \phi_{i,r+j} \\ & (j = 0, 1, 2, \dots), \end{aligned} \quad (2.9)$$

where $\phi_{i,r+j}$ denotes the corresponding terms in the Taylor-series expansions of the functions g_i . The recurrent relationships can be used to express all coefficients a_{ij} in terms of the leading coefficients a_{i0} and a_{i1} ($i = 2, 3, \dots, n$). Substituting the series $x_{i1}(x)$ into the boundary conditions (2.7), and in view of the recurrent relationships (2.9), one obtains the equations governing a_{i0}, a_{i1} :

$$\sum_{i=2}^n (R_{0i}a_{i0} + R_{1i}a_{i1}) = R_{2i} \quad (i = 2, 3, \dots, n). \quad (2.10)$$

The determinant of the system may be represented as products of an infinite number of factors [2, 22, 23]

$$\begin{aligned} K_p &= |q_{ij}|, \\ q_{ij} &= \delta_i^j [p(p-1)2\Pi(1, 0, \dots, 0) + p\Pi_x(1, 0, \dots, 0) - \Pi_{x_i x_i}(1, 0, \dots, 0)], \end{aligned} \quad (2.11)$$

where δ_i^j are Kronecker's delta, $p = 0, 1, 2, 3, \dots$.

When the generating system is linear, conditions (2.11) mean an absence of internal resonances in the generating system.

The procedure presented here is similar to that applied in a conservative case [2, 22, 23], and it is possible to make use of analogous arguments. In particular, a convergence of the series (2.8) can be proved by L'Hospital's rule.

It follows from Poincaré's theorem on the small parameter series expansion [25] that there is a value of $\varepsilon_0 > 0$ such that, for all $|\varepsilon| < \varepsilon_0$, the series in powers of ε of the functions (2.3) converge in the domain; as $\varepsilon = 0$ this solution becomes the trivial generating solution $x_{j0} = 0$ ($j = 2, 3, \dots, n$) (see the details in [2, 22, 23]).

Note that the trajectories $x_j(x)$ can be derived not only in terms of power series in x , but also by the method of iteration [2, 23].

2.3. SINGLE-DEGREE-OF-FREEDOM AUTONOMOUS SYSTEM AND A POTENTIALITY CONDITION

After having obtained a smooth NNM trajectory, the problem of finding a periodic solution reduces to the integration of the following single-degree-of-freedom autonomous system:

$$\ddot{x} + \Pi_x(x, x_2(x), \dots, x_n(x)) + \varepsilon g_1(\dot{x}, x(x), x_2(x), \dot{x}_2(x), \dots) = 0.$$

The solution $x(t)$ is determined as an inversion of the following quadrature:

$$t + \phi = \frac{1}{\sqrt{2}} \int_{x(0)}^x \frac{d\xi}{(h - \Pi(\xi, x_2(\xi), \dots) - \varepsilon g_1(\xi, \dot{x}(\xi), x_2(\xi), \dots))^{1/2}}, \quad (2.12)$$

where h is the integration constant which has the meaning of an energy constant of the conditional single-degree-of-freedom system realizable along the NNM analytical trajectory; ϕ is a phase of the solution which can be taken to be equal to zero because the original system (2.1) is autonomous.

Next, the equation of phase trajectories is of the form ($\dot{x} = v$)

$$\frac{dv}{dx} = - \frac{\Pi_x(x, x_2(x), \dots, x_n(x)) + \varepsilon g_1(x, v, x_2(x), x'_2(x)v, \dots)}{v}.$$

In the general case, one has to employ approximate methods to find all phase trajectories and due allowance is to be made for the small values of ε . However, this equation allows exact integration if g_1 is a linear function with respect to $w = v^2$.

At the analytical closed phase trajectory of the limiting cycle, the system's behavior is similar to that of a conservative system with one degree of freedom. Therefore, the condition that the work of all forces over the period is equal to zero should hold (the condition can be called a *potentiality condition*):

$$\oint g_1(x, \dot{x}(x), x_2(x), \dot{x}_2(x), \dots) dx = 0$$

or

$$\int_0^T g_1(x, \dot{x}(x), x_2(x), \dot{x}_2(x), \dots) \dot{x} dt = 0. \quad (2.13)$$

The integral is taken over the period of vibrations. Equation (2.8) is used to find the value of the energy h_0 (and, accordingly, the amplitude values of X_j) for the generating solution of the conservative system, i.e. the limiting solution (at ε tending to zero) for the required NNMs of the self-excited system (2.1).

3. Self-Excited Systems: Examples

3.1. TWO LINEARLY CONNECTED VAN DER POL OSCILLATORS

As a simple example, let us consider two linearly connected Van der Pol oscillators with nonlinear elastic characteristics:

$$\begin{aligned} \ddot{q}_1 + q_1 + \alpha q_1^3 + \beta(q_1 - q_2) - \varepsilon(1 - q_1^2)\dot{q}_1 &= 0, \\ \ddot{q}_2 + q_2 + \alpha q_2^3 + \beta(q_2 - q_1) - \varepsilon\gamma(1 - q_2^2)\dot{q}_2 &= 0. \end{aligned} \quad (3.1)$$

Here ε is a small parameter. Two vibration modes may be obtained in the zero approximation by ε : $q_2 = \pm q_1$. Let us introduce new coordinates x, y as follows:

$$q_1 + q_2 = x, \quad q_1 - q_2 = y.$$

In these coordinates, one of the NNMs, the so-called *in-phase* mode is governed by $y_0 = 0$, $x_0 = x_0(t)$ in the zeroth approximation by ε , and

$$\ddot{x}_0 = -x_0 - \frac{\alpha}{4}x_0^3, \quad \dot{x}_0^2 = 2 \left(h - \frac{x_0^2}{2} - \frac{\alpha x_0^4}{16} \right), \quad (3.2)$$

where h is the system energy constant.

An equation describing the trajectory $y(x)$ in the first approximation with respect to ε has the form

$$y_1'' \dot{x}_0^2 + y_1' \ddot{x}_0 + y_1 \left(1 + 2\beta + \frac{3\alpha}{4}x_0^2 \right) - \left(1 - \frac{x_0^2}{4} \right) \frac{\dot{x}_0}{2} (1 - \gamma) = 0. \quad (3.3)$$

The equation should be solved together with the boundary conditions of the form (2.5) (at $\dot{x}_0 = 0$, $x_0 = Q$):

$$y_1'(Q) \ddot{x}_0(Q) + y_1(Q) \left(1 + 2\beta + \frac{3\alpha}{4}Q^2 \right) = 0. \quad (3.4)$$

Substituting the solution $y(x)$ as a power series into (3.3), we obtain a set of recurrent equations in the expansion coefficients. (Note that \dot{x}_0 must be presented as a power series by x_0 too). One must also use the condition (3.4). Let $\alpha = 1.5$; $\beta = 1.5$; $\gamma = 0.5$, $\varepsilon = 0.2$. In the first approximation, one obtains the trajectory of the in-phase vibration mode:

$$y_1 \simeq -0.07203 + 0.01330x^2 - 0.000077x^4.$$

Here $x(0) = Q = 4$. The last value was obtained from the potentiality condition (2.8) in the first approximation with respect to ε in a quasiharmonic approximation.

The second of the NNMs, the so-called *out-of-phase* mode, is governed by $x_0 = 0$, $y_0 = y_0(t)$ in the zeroth approximation by ε ; respectively,

$$\ddot{y}_0 = -(1 + 2\beta)y_0 - \frac{\alpha}{4}y_0^3; \quad \dot{y}_0^2 = 2 \left(h - (1 + 2\beta)\frac{y_0^2}{2} - \frac{\alpha y_0^4}{16} \right). \quad (3.5)$$

The equation describing the trajectory $x(y)$ approximation with respect to ε has the form

$$x_1'' \dot{y}_0^2 + x_1' \ddot{y}_0 + x_1 \left(1 + \frac{3\alpha}{4}y_0^2 \right) - \left(1 - \frac{y_0^2}{4} \right) \frac{\dot{y}_0}{2} (1 - \gamma) = 0. \quad (3.6)$$

The equation should be solved together with the boundary conditions of the form (2.7) (at $\dot{y}_0 = 0$, $y_0 = Q$):

$$x_1'(Q) \ddot{y}_0(Q) + x_1(Q) \left(1 + \frac{3\alpha}{4}Q^2 \right) = 0. \quad (3.7)$$

Substituting the solution $x(y)$ as a power series into (3.6) and (3.7), one obtains, in the first approximation, the following trajectory of the out-phase vibration mode:

$$x_1 \simeq -0.10179 + 0.00881y^2 - 0.000386y^4.$$

Here $y(0) = Q = 4$. The value was obtained from the potentiality condition as above.

There is good agreement between the approximation solution and the numerical calculations checked by a computer.

3.2. PLATE VIBRATIONS IN THE FLOW OF GAS

Another example refers to the problem of plate vibrations in the flow of gas. A simple, two-degrees-of-freedom model is chosen.

Next, q_1 is the vertical displacement of the bend of the center axis of the plate; q_2 is the angle of rotation around the axis.

Both the restoring force $P(q_1)$ and the restoring moment $M(q_2)$ are nonlinear:

$$P(q_1) = (d_1 + d_2 q_1^2)q_1, \quad M(q_2) = (d_3 + d_4 q_2^2)q_2,$$

where d_1, d_2, d_3, d_4 are coefficients depending on the plate's elastic characteristics.

The aerodynamic lift is

$$A(q_2) = \sigma v^2 \sin 2q_2 \simeq \sigma v^2 \left(2q_2 - \frac{4q_2^3}{3} \right),$$

where v is the flow velocity, and σ is dependent on the flow density and the airfoil aerodynamic chord.

Let ρ be the radius of inertia with respect to the bend (peck) center axis, l is the distance from the center of rigidity to the center of mass, and m is the mass of the airfoil per unit length. The equations of motion for the system are the following:

$$\begin{aligned} m\ddot{q}_1 - ml\ddot{q}_2 + (d_1 + d_2 q_1^2)q_1 \\ = \sigma v^2 \left(2q_2 - \frac{4q_2^3}{3} \right) - ml\ddot{q}_1 + m(\rho^2 + l^2)\ddot{q}_2 + (d_3 + d_4 q_2^2)q_2 = 0. \end{aligned} \quad (3.8)$$

The analysis of the NNMs of the system (3.8) is based on the assumption that the values of d_3 and d_4 are close to those of d_1 and d_2 respectively, the sum $\rho^2 + l^2$ is close to unity and l^2 is much smaller than unity.

Make the coordinate transformation,

$$q_1 + q_2 = x\sqrt{2}, \quad q_1 - q_2 = y\sqrt{2}.$$

One obtains the following equations with respect to variables x, y in place of the system (3.8):

$$\begin{aligned} m\ddot{x} + d_1 x + 0.5d_2(x^3 + 3xy^2) \\ = \varepsilon \{ ml\ddot{x} + \sigma v^2[(x - y) - (x - y)^3/3] + 0.5m(1 - \rho^2 - l^2)(\ddot{x} - \ddot{y}) \\ + [(d_1 - d_3) + (d_2 - d_4)(x - y)^2/2](x - y)/2 \}, \\ m\ddot{y} + d_1 y + 0.5d_2(y^3 + 3yx^2) \\ = \varepsilon \{ -ml\ddot{y} + \sigma v^2[(x - y) - (x - y)^3/3] - 0.5m(1 - \rho^2 - l^2)(\ddot{x} - \ddot{y}) \\ - [(d_1 - d_3) + (d_2 - d_4)(x - y)^2/2](x - y)/2 \}, \end{aligned} \quad (3.9)$$

where ε is a formal small parameter characterizing the relative smallness of the right-hand sides of Equations (3.9). Consider the in-phase NNM close to the mode $q_1 = q_2$ or $x = x(t)$, $y = 0$ in the zeroth approximation with respect to ε . Make use of the relations (2.4) and (2.5).

The corresponding equation in the first approximation with respect to ε for the mode $y(x)$ is

$$\begin{aligned} & 2y'' \left(h_0 - \frac{d_1}{2}x - \frac{d_2}{8}x^2 \right) - y' \left(d_1x + \frac{d_2}{2}x^2 \right) + \left(d_1 + \frac{3}{2}d_2x^2 \right) y \\ & = \sigma v^2 \left(x - \frac{x^3}{3} \right) + \frac{(1 - \rho^2 - 1)^2}{2} \left(d_1x + \frac{d_2}{2}x^2 \right) \\ & \quad - \left[(d_1 - d_3) + (d_2 - d_4) \frac{x^2}{2} \right] \frac{x}{2}, \end{aligned}$$

while the boundary conditions ($\dot{x} = 0, x = Q$) may be written as

$$\begin{aligned} & \left\{ y' \left(-d_1x - \frac{d_2}{2}x^2 \right) + \left(d_1 + \frac{3}{2}d_2x^2 \right) y - \sigma v^2 \left(x - \frac{x^3}{3} \right) \right. \\ & \quad \left. + \frac{(1 - \rho^2 - l^2)}{2} \left(d_1x + \frac{d_2}{2}x^2 \right) - \left[(d_1 - d_3) + (d_2 - d_4) \frac{x^2}{2} \right] \frac{x}{2} \right\} \Big|_{x=Q}, \end{aligned}$$

h_0 being the energy of the generating system.

Numerical calculations were performed for $m = 1, l = 0.2, d_1 = 2, d_2 = 1, d_3 = 2.3, d_4 = 1.1, \sigma = 0.5, \rho = 1$, and $v^2 = \{0; 0.2; 0.4\}$.

The following expressions were obtained for the NNM $y = y(x), x(0) = Q = 1$ at $\varepsilon = 1$:

$$v^2 = 0 : \quad y \simeq 0.0374x + 0.0018x^3 - 0.00014x^5,$$

$$v^2 = 0.2 : \quad y \simeq 0.040x + 0.009x^3 - 0.0005x^5,$$

$$v^2 = 0.4 : \quad y \simeq 0.125x + 0.022x^3 - 0.001x^5.$$

The potentiality condition (2.8) for the system at hand is identically satisfied for any vibration mode defined by an analytical function $y(x)$ (or $x(y)$), and therefore the NNMs are dependent on two free parameters, namely the vibration amplitude Q and the phase ϕ .

4. Iterative Computation of Quasilinear Vibrations in Nonlinear Viscoelastic Mechanical Systems

Here we will construct periodical solutions of integro-differential equations corresponding to some viscoelastic mechanical systems. One assumes that elastic forces are dominant. A conservative system corresponding to the elastic interactions is selected as generative and the NNMs of the system are selected as generative solutions.

4.1. CONSTRUCTION OF PERIODICAL SOLUTIONS IN TWO-DEGREES-OF-FREEDOM NONLINEAR MODEL

Analysis of vibrations in nonlinear viscoelastic mechanical systems is associated with studies into integro-differential partial equations [26, 27].

By representing solutions of such equations as linear combinations of coordinate functions with time-dependent coefficients and by using the Bubnov–Galerkin method, sets of nonlinear integro-differential equations are obtained.

Restricting ourselves to two principal coordinate functions, we obtain the following set of equations with respect to the required coefficients of the linear combinations $x_1(t)$, $x_2(t)$:

$$\ddot{x}_i + \Pi_{x_i}(x_1, x_2) = \varepsilon g_i \quad (i = 1, 2), \quad (4.1)$$

where $\Pi = \Pi(x_1, x_2)$ is the potential energy that corresponds to the principal elastic interactions and is assumed to be an analytical function of its arguments; ε is a small parameter. Here and later on, the case of forced resonance will be called Case A, that of self-excited vibrations, Case B.

In Case A

$$g_i = g_{i1}(t, x_1, x_2, \dot{x}_1, \dot{x}_2) + \int_{-\infty}^t R_i(t-s) g_{i2}(x_1(s), x_2(s)) ds,$$

while in Case B

$$g_i = g_{i1}(x_1, x_2, \dot{x}_1, \dot{x}_2) + \int_{-\infty}^t R_i(t-s) g_{i2}(x_1(s), x_2(s)) ds,$$

Here, g_{i1} , g_{i2} are analytical functions which, in Case A, are continuous and periodical in t with a period of T ; R_i are relaxation kernels that satisfy the following conditions:

$$\frac{dR_i(\tau)}{d\tau} < 0, \quad \int_{-\infty}^{\infty} R_i(\tau) d\tau = 1.$$

We assume that a generating system (at $\varepsilon = 0$),

$$\ddot{x}_i + \Pi_{x_i}(x_1, x_2) = 0,$$

allows similar NNMs of the type

$$x_2 = kx_1, \quad (4.2)$$

wherein the constants k are defined by the algebraic equation

$$k\Pi_{x_i}(x_i, kx_i) = \Pi_{x_2}(x_1, kx_1).$$

Without loss of generality, we assume that after some rotation of the coordinate system (an axis of a new coordinate system is directed along the rectilinear trajectory of solution (4.2)), the generating system takes on the form

$$\ddot{x}_{i0} + \Pi_{x_i}(x_{i0}, x_{10}) = 0, \quad x_{20} = 0,$$

i.e., $k = 0$ for the similar NNMs under consideration and the generating solution in Case A is

$$x_{i0} = x_{i0}(F, \Phi, t + \phi), \quad x_{20} = 0, \quad (4.3)$$

in Case B,

$$x_{i0} = x_{i0}(F, \Phi, t), \quad x_{20} = 0, \quad (4.4)$$

where F and Φ are two amplitudes of vibrations (two values of x_{10} that correspond to the cusp, $\dot{x}_{10} = 0$); ϕ is the generating solution phase which, in Case B, is arbitrary due to the system being autonomous and therefore is not explicitly introduced into the solution. The amplitudes F and Φ are related to the period of the generating solutions (4.3) and (4.4) by the equality

$$T(F, \Phi) = \sqrt{2} \int_{\Phi}^F \frac{dx}{(h - \Pi(x, 0))^{1/2}},$$

where h is the energy of the generating solution,

$$h = \Pi(F, 0) = \Pi(\Phi, 0).$$

Without loss of generality, one assumes that F is greater or equal to Φ .

Let us look for a solution of the original perturbed system (4.1) in the form

$$x_1 = x_{10} + \xi, \quad x_2 = \eta.$$

In order to find ξ and η , we obtain from (4.1), selecting a linearized part of the system with respect to ξ and η :

$$\ddot{\xi} + p_1(x_{10})\xi = G_1, \quad \ddot{\eta} + p_2(x_{10})\eta = G_2, \quad (4.5)$$

where

$$p_1(x_{10}) = \Pi_{x_1 x_1}(x_{10}, 0), \quad p_2(x_{10}) = \Pi_{x_2 x_2}(x_{10}, 0);$$

$$G_1 = \varepsilon g_1 - \Pi_{x_1} - p_1(x_{10})\xi, \quad G_2 = \varepsilon g_2 - \Pi_{x_2} - p_2(x_{10})\eta,$$

here $g_1, g_2, \Pi_{x_1}, \Pi_{x_2}$ are calculated as $x_1 = x_{10} + \xi, x_2 = \eta$.

It is clear that the first equation of (4.5), linearized with respect to ξ and η , is a variational equation for the generating solution. Therefore, fundamental solutions of the linearized system can be obtained by the derivation of the generating solution with respect to the arbitrary parameters, namely the amplitude F and the phase ϕ [15]:

$$w_{11} = \frac{1}{\omega} \frac{dx_{10}}{d\phi} = \gamma \dot{x}_{10} \quad (\gamma = \text{const.}), \quad w_{12} = \frac{1}{\omega'} \frac{dx_{10}}{dF},$$

where

$$\omega = \frac{2\pi}{T(F)}, \quad \omega' = \frac{d\omega}{dF}.$$

The first fundamental solution is periodical, while the second one involves a product of some periodical function by t .

A fundamental solutions of the second linearized equation in (4.5) may be obtained for certain classes of potential functions Π in the form of hypergeometric functions or Lamé functions [22, 23, 28]. In other cases, it may be constructed by a power series expansion in x_{10} or by other methods, see [2, 23]. Let us denote the fundamental solutions by w_{21}, w_{22} .

4.2. ITERATION PROCESS

In order to find periodical quasnormal solutions of Equations (4.5), we employ here the following iteration process:

$$\begin{aligned}
\xi_{k+1} &= \frac{1}{\Delta_1} \int_0^t G_{1k}^\tau (w_{11}^\tau w_{12}^t - w_{12}^\tau w_{11}^t) d\tau + \frac{1}{T\Delta_1} \left(w_{12} - w_{11} \frac{\dot{w}_{12}^0}{\dot{w}_{11}^0} \right) \int_0^t G_{1k}^\tau w_{12}^\tau d\tau, \\
\eta_{k+1} &= \frac{1}{\Delta_2} \int_0^t G_{2k}^\tau (w_{21}^\tau w_{22}^t - w_{22}^\tau w_{21}^t) d\tau + D_{1k} w_{21} + D_{2k} w_{22}, \\
\dot{\xi}_{k+1} &= \frac{1}{\Delta_1} \int_0^t G_{1k}^\tau (w_{11}^\tau \dot{w}_{12}^t - w_{12}^\tau \dot{w}_{11}^t) d\tau + \frac{1}{T\Delta_1} \left(\dot{w}_{12} - \dot{w}_{11} \frac{\dot{w}_{12}^0}{\dot{w}_{11}^0} \right) \int_0^t G_{1k}^\tau w_{12}^\tau d\tau, \\
\dot{\eta}_{k+1} &= \frac{1}{\Delta_2} \int_0^t G_{2k}^\tau (w_{21}^\tau \dot{w}_{22}^t - w_{22}^\tau \dot{w}_{21}^t) d\tau + D_{1k} \dot{w}_{21} + D_{2k} \dot{w}_{22}, \tag{4.6}
\end{aligned}$$

Here

$$\Delta_i = \begin{vmatrix} w_{i1} & w_{i2} \\ \dot{w}_{i1} & \dot{w}_{i2} \end{vmatrix} = \text{const.}; \quad f_{ij}^\theta(z) = f_{ij}(z)|_{z=\theta};$$

$\xi_k^{(n)}$ ($n = 1, \dots, 4$) are substituted into the right-hand sides of the above relationships (4.6) together with ϕ_k (Case A) or F_k (Case B); D_{ik} are arbitrary constants.

Let us use the conditions of T -periodicity of the functions η_{k+1} , $\dot{\eta}_{k+1}$, to find the unknown constants D_{1k} , D_{2k} (the period T is given in Case A and unknown in Case B):

$$\begin{aligned}
&D_{1k}(w_{21}^T - w_{21}^0) + D_{2k}(w_{22}^T - w_{22}^0) \\
&= -\frac{w_{21}^T}{\Delta_2} \int_0^T G_{2k}^\tau w_{22}^\tau d\tau + \frac{w_{22}^T}{\Delta_2} \int_0^T G_{2k}^\tau w_{21}^\tau d\tau, \\
&D_{1k}(\dot{w}_{21}^T - \dot{w}_{21}^0) + D_{2k}(\dot{w}_{22}^T - \dot{w}_{22}^0) \\
&= -\frac{\dot{w}_{21}^T}{\Delta_2} \int_0^T G_{2k}^\tau w_{22}^\tau d\tau + \frac{\dot{w}_{22}^T}{\Delta_2} \int_0^T G_{2k}^\tau w_{21}^\tau d\tau. \tag{4.7}
\end{aligned}$$

Besides: the periodicity conditions for ξ_{k+1} and $\dot{\xi}_{k+1}$ must be written in Case A:

$$\begin{aligned}
&\int_0^T G_{1k}^\tau w_{11}^\tau d\tau = 0, \\
&T = T(F, \Phi) = \sqrt{w} \int_\Phi^F \frac{dx}{(\Pi(F, 0) - \Pi(x, 0))^{1/2}} \tag{4.8}
\end{aligned}$$

(the period T is known); $\Pi(F, 0) = \Pi(\Phi, 0)$.

From this set, the phase ϕ may be found in the corresponding approximation of the perturbation method, together with the amplitudes F, Φ of the generating solution.

In Case B, the periodicity conditions for ξ_{k+1} and $\dot{\xi}_{k+1}$ have a slightly different form:

$$\int_0^{T_k} G_{1k}^\tau w_{11}^\tau d\tau = 0,$$

$$T = T_k = \sqrt{w} \int_{\Phi_k}^{F_k} \frac{dx}{(\Pi(F_k, 0) - \Pi(x, 0))^{1/2}} \quad (4.9)$$

(the period T_k is not known); $\Pi(F_k, 0) = \Pi(\Phi_k, 0)$.

Equations (4.9) may evaluate the period T_k and the amplitudes F_k, Φ_k of generating solution in the corresponding approximation with respect to ε .

It is assumed that all variables and parameters vary over a limited region, all functions in (4.1) and their derivatives, inclusive of those of third-order, are bounded, satisfy the Lipschitz condition, and $w_{ij}, \dot{w}_{ij}, \Delta_{ij}$ ($i, j = 1, 2$) are also bounded; the remaining conditions are given in [29, 30].

For the iterations of (4.6) to converge to the solution of the initial integro-differential equations, the following principal conditions are necessary: the variables $\xi_{k+1}, \eta_{k+1}, \dot{\xi}_{k+1}, \dot{\eta}_{k+1}$ should remain in a bounded region as should the phase ϕ_k and the amplitudes F and Φ in Case A or the period T_k and the amplitudes F_k, Φ_k in Case B. The operators that determine the transformation (4.6) should be contracting; the periodicity conditions (4.8) or (4.9) should be uniquely solvable with respect to ϕ_k, F, Φ or T_k, F_k, Φ_k ; and the corresponding operators determining the phase, amplitude, and period of the vibrations should be contracting.

These requirements lead to a set of inequalities that make the small parameter ε bounded above [29, 30]; the inequalities are cumbersome and therefore are not shown here.

In the first approximation of the iteration process, the number of periodic solutions of the set of integro-differential equations is determined by the number of roots of Equations (4.8) in Case A or the number of roots F_k of Equations (4.9) in Case B. When ε tends towards zero, all these solutions tend towards NNMs of the generating conservative (elastic) system.

4.3. RESONANCE NONLINEAR TRANSVERSAL VIBRATIONS OF A VISCOELASTIC ROD

By way of example, let us consider resonance nonlinear transversal vibrations of a viscoelastic rod. Within the framework of the well-known Kirchhof hypothesis, the equations of motion are taken in the form

$$u_{tt} + EIu_{xxxx} - \frac{EI}{2F} \int_0^l (u_\xi)^2] r m d\xi u_{xx}$$

$$= \varepsilon \lambda \sin pt \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{l} + \varepsilon \int_{-\infty}^t R(t-s) u^3(s) ds,$$

where u is the transversal displacement; x is the longitudinal independent variable; E is the Young modulus; I is the moment of inertia, l is a distance between end points, $R(z)$ is the

relaxation kernel, F is an area of the transversal cross-section of the rod, and ε is a small parameter. The rod ends are assumed to be restrained.

Setting the spatial wave formation in the form (m and n are integers)

$$u = x_1(t) \sin \frac{m\pi x}{l} + x_2(t) \sin \frac{n\pi x}{l},$$

by using the standard Bubnov–Galerkin method one obtains:

$$\begin{aligned} \ddot{x}_1 + \frac{EF}{4} \left[\left(\frac{m\pi}{l} \right)^4 x_1^3 + \left(\frac{m\pi}{l} \right)^2 \left(\frac{n\pi}{l} \right)^2 x_1 x_2^2 \right] \\ = -EI \left(\frac{m\pi}{l} \right)^4 x_1 + [\sin pt + J(x_1, x_2)], \\ \ddot{x}_2 + \frac{EF}{4} \left[\left(\frac{n\pi}{l} \right)^4 x_2^3 + \left(\frac{m\pi}{l} \right)^2 \left(\frac{n\pi}{l} \right)^2 x_2 x_1^2 \right] \\ = -EI \left(\frac{n\pi}{l} \right)^4 x_2 + \varepsilon [\sin pt + J(x_2, x_1)]. \end{aligned}$$

Here

$$J(u, v) = \int_{-\infty}^t R(t-s) \left(\frac{3}{4} u^3(s) + \frac{3}{2} uv^2 \right) ds, \quad \lambda a_n = \lambda a_m = 1.$$

In the zeroth approximation, $\varepsilon = 0$, there are two nonlinear normal vibration modes with rectilinear trajectories (similar NNMs), $x_1 = 0$ and $x_2 = 0$. A numerical computation using the presented analytical approach was performed by Petrov for a rod of acrylic plastic ST-1 having $E = 6.3 \times 10^9$ N m⁻² and $l = 0.4$ m; the rod had a square cross-section with a side of 0.02 m; the representation $R(\tau) = A e^{-\beta\tau} / \tau^{1-\alpha}$, with $A = 0.0286$, $\beta = 0.05$ and $\alpha = 0.075$ was assumed for the relaxation kernel; $n = 1$, $m = 2$.

Parameters of the resonance modes close to the mode $x_2 = 0$ were calculated in the zeroth and the first approximations with respect to ε . Some variation of the analytical procedure was introduced here, namely, the amplitude F_0 of the zeroth approximation was given and the period T was calculated.

When the amplitude F_0 of the zeroth approximation is equal to 5, the period $T = 1.916 \times 10^{-3}$ s and the phase shift between the zeroth and the first approximation solutions, x_{10} and x_{11} is: $\pi/2 < \phi < \pi$. When $F_0 = 8$, the calculations yield $T = 1.197 \times 10^{-3}$ s and the phase shift between x_{10} and x_{11} lies within the above range.

5. Conclusions and Discussion

At the present time, it is known that NNMs are typical periodical solutions in n -degrees-of-freedom nonlinear conservative systems. Moreover, normal or quasilinear vibrations exist in broad classes of nonlinear, near-conservative systems, as was indicated above, in nonautonomous systems (forced resonances), self-excited systems and systems containing viscoelastic interactions which are presented in the corresponding equations in a form of integral operators. It is found that approaches for the analysis of NNMs in conservative systems are applicable to near-conservative nonlinear finite-dimensional systems.

It is fruitful to extend the concept of NNMs to distributed nonlinear systems. One of possible extensions of the concept to continuous oscillators was performed by Vakakis [22]. In the spirit of Rosenberg [3, 4] and Shaw and Pierre [11], the continuous nonlinear normal modes were defined as motions during which all material points of the system vary equiperiodically, vanishing or reaching their extremal values at the same instant in time. This way leads to equations which are similar to the equations considered previously in the case of conservative or near-conservative systems.

Other generalizations of the NNMs to distributed systems are possible too:

1. It is possible to determine NNMs in distributed systems as solutions in separated variables with a uniform time function. The solutions correspond to similar normal modes in finite-dimensional systems with rectilinear trajectories in the configuration space.
2. During NNM, a finite-dimensional system behaves like a single-DOF conservative one. Therefore it is advisable to introduce nonlinear stationary traveling waves with a single phase (these are so-called *simple waves*) as a new generalization of NNMs to the distribution case.
3. There is a coupling of the existence of NNMs and symmetry properties of finite-dimensional systems [22]. It is possible to determine NNMs in a distributed system as solutions which are invariant with respect to the symmetry groups of the system.

Note, finally, that problems of NNMs extending to distributed nonlinear systems as a development of approaches for the analysis of the solutions, are interesting, but no details are available at present.

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