Yuri V. Mikhlin
National Technical University "KhPI,"
Kharkov 61002, Ukraine
e-mail: muv@kpi.kharkov.ua

Konstantin V. Avramov
National Technical University "KhPI,"
Kharkov 61002, Ukraine;
A.N. Podgorny Institute for
Mechanical Engineering Problems,
National Academy of Sciences of Ukraine,
Kharkov 61046, Ukraine
e-mail: kvavr@kharkov.ua

Nonlinears Normal Modes for Vibrating Mechanical Systems. Review of Theoretical Developments

Two principal concepts of nonlinear normal vibrations modes (NNMs), namely the Kauderer–Rosenberg and Shaw–Pierre concepts, are analyzed. Properties of the NNMs and methods of their analysis are presented. NNMs stability and bifurcations are discussed. Combined application of the NNMs and the Rauscher method to analyze forced and parametric vibrations is discussed. Generalization of the NNMs to continuous systems dynamics is also described. [DOI: 10.1115/1.4003825]

1 Introduction

Nonlinear normal vibrations modes (NNMs) are periodic motions of specific type, which can be observed in different nonlinear mechanical systems. NNMs can be applied to analyze free, forced, parametric, and self-sustained vibrations. NNMs are obtained in wide classes of systems, using in mechanical engineering. In particular, NNMs are observed in models of pretwisted beams, cylindrical shells, including shells interacting with fluids, variable thickness shallow shells with complex base, rotors, etc. Effective analytical, semi-analytical, and numerical methods for analysis of the NNMs are developed for wide set of mechanical systems.

NNMs are a generalization of normal (or principal) vibrations of conservative linear systems. In the normal vibration mode, a finite degree-of-freedom system vibrates like a single-degree-of-freedom conservative one.

There is a remarkable example of linear normal vibrations used as generative solutions to construct periodic solutions of quasilinear systems. Lyapunov showed [1] that nonlinear finite-dimensional systems with the first integral have the one-parameter family of periodic solutions, which tend to normal vibration modes of linear systems if amplitudes tend to zero. These systems are named after Lyapunov. Lyapunov used a power series in amplitude with time periodic coefficients and also series by two phase coordinates to construct such solutions. These periodic solutions can be obtained by construction of modal lines in a configuration space too; it will be described in Sec. 2.

Kauderer [2] was the first who developed quantitative methods for the NNMs analysis in two-DOF conservative nonlinear systems. Existence of periodical motions similar to the NNMs in Hamiltonian systems was considered by Seifert [3]. These studies were continued by Rosenberg and others researchers [4–9]. Rosenberg considered *n*-DOF conservative systems and deduced the first definition of NNM as "vibrations in unison," i.e., synchronous periodic motions, where all material points of the system reached their maximum and minimum values at the same instant of time; hence, the NNM is represented by either a straight modal line ("similar NNM") or a modal curve ("nonsimilar NNM") in the configuration space of the system. He considered wide classes of essentially nonlinear systems, which have nonlinear vibrations modes with straight modal lines. This is a direct generalization of the linear normal modes to a nonlinear case. The similar NNMs do not depend on the system energy. For example, "homogeneous

systems" with a potential, which is an even homogeneous function of the general coordinates, belong to such a class. The NNMs based on determination of modal lines (trajectories) in configuration space can be called the Kauderer–Rosenberg nonlinear normal modes. Development of this concept, the NNMs construction and an analysis of their stability and bifurcations, was published in different papers, which will be analyzed in the next sections of this paper.

Similar NNMs are not typical in nonlinear systems. In general, the NNM modal lines in a configuration space are curvilinear. Curvilinear modal lines of several dynamical systems are studied in Refs. [10,11]. The power-series method was proposed to construct the curvilinear trajectories in the conservative systems by Manevich and Mikhlin [12–14]. The localized nonlinear modes are studied in Ref. [15]. In this case, one or few particles, which are disposed adjacently, have large vibrations amplitudes and other particles have small amplitudes. The global dynamics of the nonlinear systems near NNMs is analyzed using the Poincaré map in the paper [16]. Padé approximations [17] are used to derive the NNMs with arbitrary amplitudes.

Shaw and Pierre [18–21] developed an alternative concept of NNMs for nonlinear finite-DOF systems. Their researches are based on the computation of invariant manifolds of motion, where the system oscillations take place. The similar approach was suggested by Lyapunov [1] to construct periodic solutions in the system phase space. This second type of the NNMs is called the Shaw–Pierre nonlinear normal modes.

Note that publications on the NNMs of nonconservative systems are not numerous. The periodic motions in nonautonomous systems close to Lyapunov ones are analyzed by Malkin [22]. The Rauscher method is used to analyze the normal vibrations modes of nonautonomous systems in Ref. [23]. Rauscher's approach and the power-series method for the modal lines construction are used to derive resonance solutions in Ref. [24]. NNMs of the self-excited systems close to conservative ones are analyzed in Ref. [25]. NNMs of parametric vibrations are described in Ref. [26]. Other results on the NNMs in nonconservative systems will be presented in Sec. 7.

There are other generalizations of the NNMs. One of them is associated with symmetry of a dynamical system. Such symmetry as a cause of existence of NNMs with rectilinear modal lines is investigated by using the group theory in Refs. [15, 27, 28]. Later the analysis of symmetric dynamical systems is extended by the concept of bushes of normal modes, which is considered in the papers [28,29]. The symmetry characteristics and the NNMs existence are discussed in Sec. 4.

NNMs in systems with nonsmooth characteristics are analyzed in papers [30,31], where the saw-tooth time transformation is used. NNMs of the systems with gyroscopic forces are considered in papers [32,33]. A generalization of the NNMs, when all generalized coordinates are functions of a single scalar phase, is presented in Ref. [34].

Generalization of the NNMs to continuous systems is made in the papers [18,35,36]. Two different approaches for the continuous systems are suggested by King and Vakakis [35] and by Shaw and Pierre [18]. Theoretical basis of the Shaw–Pierre nonlinear modes for continuous system is an extension of center manifolds to distributed systems, which is stated in Ref. [37].

NNMs have been used to solve applied problems of mechanical and aerospace engineering. Such vibrations take place in structures and machines. NNMs of free vibrations of cylindrical shells are considered in papers [33,38]. NNMs of shallow shells with geometrical nonlinearity interacting with a fluid are analyzed in Ref. [39]. The absorption of free and forced vibrations of finite-DOF mechanical systems is treated by using NNMs in Refs. [40–45]. NNMs of shallow arches snap-through motions are considered in Ref. [46], where the Ince algebraization is used to study a stability of such motions. NNMs are used to analyze dynamics of pretwisted beam with geometrical nonlinearity in Refs. [38,47]. NNMs of beams parametric vibrations are analyzed in the paper [26]. R-function method and NNMs are applied jointly to analyze nonlinear free vibrations of shallow shells with complex base in the paper [47]. NNMs of eight-degree-of-freedom models of parametrically excited cylindrical shells are considered in the paper [48]. NNMs of self-oscillations of rotors in short journal bearings are studied in the papers [49,50]. Nonlinear suspension dynamics is considered in Ref. [51]. The next review paper by the same authors will be devoted to engineering applications of NNMs.

Energy transfer in essentially nonlinear systems is investigated previously in numerous publications by Manevitch, Vakakis, Gendelman, Bergman, Lamarque, Pilipchuk et al. [40,41,52–59]. An important part of these works is an analysis of the "passive irreversible transfer of energy" (or "the energy pumping") from the main substructure to light auxiliary nonlinear attachment. It may be treated as transition of the motions to localized nonlinear normal mode. This problem is relative to localization of energy and vibrations absorption in engineering systems. Conception of NNMs, the multiple-scale method, method of averaging, the method of complex variables suggested by Manevitch, and other approaches are used to study these problems. These problems are not considered in this review.

Basic results on NNMs are presented in Refs. [14,15,60], which describe quantitative and qualitative analyses of NNMs in conservative and nonautonomous systems, including localized modes, stability, and bifurcation analysis. The main concept of NNMs construction and various applications are treated in a review paper of Vakakis [61]. The paper [62] illustrates in a simple manner the foundations of NNMs.

This paper is organized as follows. In Sec. 2, the Kauderer–Rosenberg concept of the NNMs is presented. Equations and boundary conditions for modal lines in a configuration space of the finite-DOF conservative systems are described. Properties of the NNMs are also treated. Matching of the local expansions of NNMs by using Pade' approximants is considered in Sec. 3. Section 4 presents a use of the group-theoretical approach for computation of NNMs. Analysis of the NNMs stability and bifurcations is made in Sec. 5. Concept of the Shaw–Pierre NNMs is considered in Sec. 6. Approaches for the NNMs analysis in nonconservative systems are presented in Sec. 7. Numerical methods for the NNMs calculations are treated in Sec. 8. Development of the NNMs concepts to continuous systems is considered in Sec. 9. The Shaw–Pierre and King–Vakakis concepts for the continuous systems are also presented. Localization of the NNMs is analyzed in Sec. 10. Results on the NNMs qualitative theory are presented in Sec. 11.

2 Kauderer–Rosenberg NNMs

2.1 Modal Lines in Configuration Space. NNMs of the conservative systems can be depicted by modal lines in configuration space. The finite-DOF conservative system with a single trivial equilibrium is considered; the equations of the system motions are the following:

$$m_i \ddot{x}_i + \Pi_{x_i} = 0 \tag{2.1}$$

where $\Pi_{x_i} = \partial \Pi / \partial x_i$; $\Pi = \Pi(x)$ is the system potential energy, which is assumed as the positive definite analytical function. The next change of the variables is used: $\sqrt{m_i} x_i \rightarrow x_i$. Then the energy integral of the system is presented as

$$\frac{1}{2} \sum_{k=1}^{n} \dot{x}_k^2 + \Pi(x_1, x_2, ..., x_n) = h \tag{2.2}$$

where h is a value of the system energy. Note that all motions are bounded by closed equipotential surface $\Pi(x_1, ..., x_n) = h$, where the relations $\dot{x}_1 = \dot{x}_2 = \cdots = \dot{x}_n = 0$ are satisfied.

The equations of modal lines in a configurational space can be obtained as the Euler equations for the Jacobi variation principle in the following form [63]:

$$\delta S = \delta \int_{P_1}^{P_2} \sqrt{2(h - \Pi)} ds = 0 \tag{2.3}$$

where $ds^2 = \sum_{k=1}^{n} dx_k^2$. This variation principle can be presented in the following form [63]: $\delta S = \delta \int_{\alpha(P_1)}^{\alpha(P_2)} \sqrt{2(h - \Pi)} \sqrt{\sum_{k=1}^{n} (dx_k/d\alpha)^2} d\alpha = 0$, where α is an arbitrary parameter.

In order to obtain NNMs, some arbitrary generalized coordinate is chosen as the independent variable. For example, it is possible to take: $x \equiv x$. Then NNM modal line in a configuration space can be presented as

$$x_i = p_i(x) \quad (i = 2, 3, ..., n) \tag{2.4}$$

where $p_i(x)$ are single-valued analytic functions. The relations (2.4) define the Kauderer–Rosenberg NNMs, which are called "vibrations in unison." These NNMs describe synchronous periodic motions when all general coordinates vibrate equiperiodically. The dynamical system (2.1) can be rewritten with respect to the new independent variable x, using the following relations:

$$\frac{d}{dt} = \dot{x} \frac{d}{dx} ; \; \frac{d^2}{dt^2} = \dot{x}^2 \frac{d^2}{dx^2} + \ddot{x} \frac{d}{dx} \tag{2.5}$$

Excluding $\dot{x}^2$ from the energy integral (2.2), it is derived the following:

$$\dot{x}^2 = 2(h - \Pi) \bigg/ \left(1 + \sum_{k=2}^{n} x_k'^2 \right) \tag{2.6}$$

where $x_k' = dx_k/dx$. Using the relations (2.2)–(2.6), it is possible to obtain the following equations of modal lines in the system configuration space:

$$2x_i'' \frac{h - \Pi}{1 + \sum_{k=2}^{n} x_k'^2} - \Pi_x x_i' = -\Pi_{x_i} \; (i = 2, 3, ..., n) \tag{2.7}$$

Note that these equations can also be obtained from the Jacobi variation principle. Although a number of the equations (2.7) is smaller by one than a number of the equations (2.1), the equations
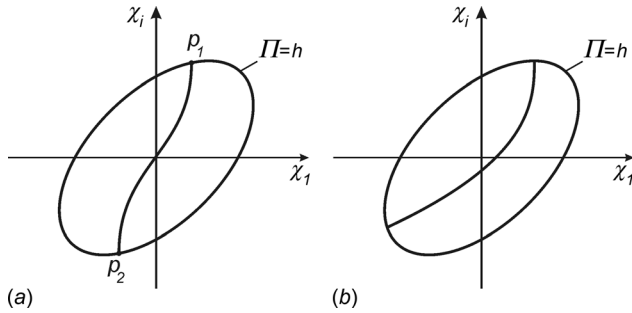
Fig. 1 Curvilinear modal lines of NNM in a configuration space: (a) the trajectory pass through the origin; (b) the trajectory does not pass through the origin

(2.7) are nonautonomous and nonlinear, even for linear conservative systems. Moreover, these equations have singular points on the maximal equipotential surface. However, these equations are convenient to obtain rectilinear and nearly rectilinear modal lines of the NNMs [9,12–15].

The modal lines reach the maximum equipotential surface $\Pi(x_1, ..., x_n) = h$. An analytical continuation of these trajectories to the equipotential surface is possible. The following boundary conditions guarantee the analytical continuation [6–9,12–15]

$$x_i'(X)[-\Pi_x(X, x_2(X)..., x_n(X))] = -\Pi_{x_i}(X, x_2(X)..., x_n(X))$$
$$(i = 2, 3..., n) \qquad (2.8)$$

where X is the amplitude of vibration; X, $x_2(X), ..., x_n(X)$ are the cusps of modal lines lying on the maximal equipotential surface (Fig. 1), which satisfy the following equation:

$$\Pi(X, x_2(X), ..., x_n(X)) = h \qquad (2.9)$$

Moreover, the Eq. (2.8) defines the orthogonality conditions of the modal lines to the maximum equipotential surface.

The modal line (2.4) can be obtained from the Eq. (2.7) and the boundary conditions (2.9). Then the time dependent motion on the NNMs will be determined from the first equation of the system (2.1), which can be presented as

$$\ddot{x} + \Pi_x(x) = 0 \qquad (2.10)$$

where $\Pi_x(x) = d\Pi/dx = \Pi_x(x, p_2(x), ..., p_n(x))$. The periodic solution $x(t)$ can be obtained from the Eq. (2.10) in the following integral form:

$$t + \varphi = \frac{1}{\sqrt{2}} \int_X^x \frac{d\xi}{\sqrt{h - \Pi(\xi)}} \qquad (2.11)$$

Thus, the NNMs are two-parametric (h, φ) family of periodic solutions with smooth modal lines in configuration space. Note that the energy and the amplitude X of the single-DOF nonlinear oscillator are connected by the Eq. (2.9). Condition of closure of all equipotential surfaces for different values of energy h and absence of equilibrium positions besides $x_i = 0$ $(i = 1, 2, ..., n)$ guarantee a solvability of the Eq. (2.9) [12–15]. If the power series expansion of the potential energy $\Pi(x)$ has only terms with the even power, two amplitude values X_i $(i = 1, 2)$ satisfies the following relations: $X_1 = -X_2$.

2.2 Properties of NNMs. Rosenberg [5–9] determined a wide class of essentially nonlinear conservative systems having rectilinear NNMs in a configuration space

$$x_i = k_i x \quad (k_i = \text{const}) \qquad (2.12)$$

where k_i are modal constants. For example, systems, which potential energy is the even homogeneous function of the general coordinates, belong to such class. Using the Lusternik and Shnirelman theory on the nonlinear eigenvalues problem [64], van Groesen shows [65] that the homogeneous systems with n degree-of-freedom have not less than n nonlinear normal modes.

The nonlinear systems with polynomial potential energy were considered by Atkinson et al. [66]. They derived the general conditions for potential energy, which guarantee an existence at least one rectilinear NNMs. Kauderer and Rosenberg [2,9] showed that the rectilinear NNMs $x_i = k_i x$ $(k_i = \text{const})$ intersect orthogonally all equipotential surfaces. Then the Eq. (2.8) is reduced to the next form: $\Pi_x(x, x_2, ..., x_n) x_i' = \Pi_{x_i}(x, x_2, ..., x_n)$ $(i = 2, 3, ..., n)$. These are conditions of orthogonality of the rectilinear modal lines to the equipotential surfaces $\Pi(X, x_2(X), ..., x_n(X)) = h$.

The number of NNMs in the nonlinear case can exceed the number of the system degrees-of-freedom. This remarkable property has no analogy in linear systems, except several degenerate cases. As an example, the following system is considered:

$$\ddot{x}_1 + x_1 + x_1^3 + \gamma(x_1 - x_2)^3 = 0$$
$$\ddot{x}_2 + x_2 + x_2^3 + \gamma(x_2 - x_1)^3 = 0 \qquad (2.13)$$

The rectilinear NNMs $x_2(t) = c\,x_1(t)$ are determined from the following algebraic equation with respect to the modal constant c:

$$c(1 - c^2) + \gamma(1 - c)^3(1 + c) = 0 \qquad (2.14)$$

Equation (2.14) has two solutions $c_{1,2} = \pm 1$ for arbitrary value of γ, which correspond to in-phase and out-of-phase NNMs. For $\gamma < 0.25$, the Eq. (2.14) possesses two other real roots $c_{3,4} = 0.5\left[-\gamma^{-1} + 2 \pm \sqrt{\gamma^{-1}(\gamma^{-1} - 4)}\right]$. These real roots correspond to two NNMs, which bifurcate from the out-of-phase NNM at $\gamma = 0.25$; the out-of-phase mode becomes unstable. So, if $\gamma < 0.25$, four NNMs exist in the nonlinear system (2.13).

Rosenberg showed that for several specific cases the general solution of the nonlinear system can be obtained by superposition of the NNMs [9,67]. Principal aspects of the Rosenberg's theory are presented in a book by Blaquiere [68].

2.3 Analysis of Curvilinear Modal Lines. It seems that Rosenberg and Kuo [10] are the first who analyzed curvilinear modal lines of the NNMs for finite-DOF systems. Following the papers [12–15], the curvilinear modal lines can be determined in the form of power series. Now this procedure is treated for the following dynamical system:

$$\ddot{q}_i + \Pi_{q_i}^{(0)}(q_1, ..., q_n) + \varepsilon\,\Pi_{q_i}^{(1)}(q_1, ..., q_n) = 0;\ i = 1, ..., n \quad (2.15)$$

where $\varepsilon << 1$; $\Pi^{(0)} + \varepsilon\,\Pi^{(1)}$ is a system potential energy. It is assumed that the unperturbed system $(\varepsilon = 0)$ has rectilinear NNMs of the form $q_i = k_i q_1$; $i = 2, ..., n$. After the coordinate rotation, the NNM of the unperturbed system can be presented as

$$x_i = 0;\ (i = 2, ..., n)\,;\ x \equiv x_1(t) \qquad (2.16)$$

Then the system (2.15) can be rewritten as

$$\ddot{x}_i + \tilde{\Pi}_{x_i}^{(0)}(x_1, ..., x_n) + \varepsilon\,\tilde{\Pi}_{x_i}^{(1)}(x_1, ..., x_n) = 0;\ i = 1, ..., n \quad (2.17)$$

The curvilinear NNMs are determined as power series by the small parameter ε

$$x_i = \sum_{k=1}^{\infty} \varepsilon^k x_{ik}(x) \qquad (2.18)$$

The functions $x_{ik}(x)$ can be represented as the following power series by x:

$$x_{ik} = \sum_{l=1}^{\infty} a_{ik}^{(l)} x^l \qquad (2.19)$$

The method of the power series (2.19) determination is treated in Refs. [12–15, 69]. Equation (2.6) and the boundary conditions (2.8) are decomposed to corresponding approximations by the small parameter. Coefficients of the power series (2.19) in every approximation by ε are determined from the systems of linear algebraic equations, which guarantee a solvability and uniqueness of the solution. This uniqueness is violated in the cases of internal resonances in the unperturbed system.

If the potential energy of the unperturbed system (2.17) $\tilde{\Pi}^{(0)}(x, x_2, ..., x_n)$ is the even homogeneous function with respect to all generalized coordinates of degree $r + 1$, the conditions of existence of the unique NNM close to the rectilinear NNM (2.16) are the following [12–15]:

$$K_p \neq 0 \qquad (2.20)$$

where $K_p = Det(Q^{(p)})$; $Q^{(p)} = \|q_{ij}^{(p)}\|$; $p = 0, 1, ...$; $q_{ij}^{(p)} = \delta_{ij} [p(p-1)2\tilde{\Pi}^{(0)}(1, 0, ..., 0) + p\tilde{\Pi}_x^{(0)}(1, 0, ..., 0) - \tilde{\Pi}_{x_i x_i}^{(0)}(1, 0, ..., 0)]$; and δ_{ij} is the Kronecker symbol.

If the unperturbed system is linear, the conditions (2.20) eliminate the internal resonances in this system.

2.4 Development of Kauderer–Rosenberg Concept. Yang and Rosenberg [70] considered the NNMs of particle with two translational degrees of freedom. This particle is attached to fixed points by two identical springs. The principle of least action is used to construct the NNMs in Refs. [71,72]. Minimal normal modes (MNMs) are defined as nonlinear normal modes that give a minimum to the Jacobi's principle. It is shown that for a certain class of two-DOF nonlinear conservative systems, a pair of MNMs generically occurs. Both the generic and nongeneric bifurcations of MNMs are obtained. A similar definition of the NNMs as geodesic curves in Riemannien manifold is made in Ref. [73]. This manifold is determined by the Lagrange–Jacobi variational principle. The NNMs can be obtained from the variational principle in the form of Taylor series.

The normal modes of the nonlinear system with two-DOF, which contains either hard or soft springs, are investigated in Ref. [74]. It is obtained three vibration modes, one of which does not have any counterpart in the linear theory.

Description of NNMs in terms of solutions of initial value problems is developed for the nonlinear dual-mass system in Ref. [75]. An existence of such NNMs and their dependence on the energy is considered. In Ref. [76], the asymptotic methodology and the power series are used to construct NNMs in several nonlinear systems. NNMs of two-DOF nonlinear system with three nonlinear springs are considered in a configuration space by Happawana et al. [77]. Nonsimilar NNMs of coupled nonlinear oscillators with quadratic and cubic nonlinearities are analyzed using the asymptotic methodology from the Sec. 2.3 in Ref. [78].

It is shown in Ref. [79] that the NNMs can exist only in intermediate amplitude range and vanishing by coalescence in limiting points. This result is obtained for rectilinear approximation of the NNMs in a conservative system with the potential energy, which involves terms of the second, fourth, and sixth powers of general coordinates. The harmonic balance method is used to construct the NNMs by Szemplinska-Stupnicka [80]. Perturbation techniques and harmonic balance method are used to calculate NNMs by Pak [81]. The NNMs and their bifurcations in nonholonomic systems are studied by the averaging method in Ref. [82]. NNMs in the case of the internal resonances are considered in the paper [83]; properties of NNMs are connected with the global dynamics of the system. Both localized and nonlocalized NNMs are analyzed.

Different asymptotic methods are used to construct NNMs of two-DOF spring–mass–pendulum system in Ref. [84]. The numerical methodology, which will be described in Sec. 8, is developed to calculate the NNMs; this method is used to study the NNM bifurcations in dependence of linear frequencies ratio and total energy. The NNMs of the conservative spring–mass–pendulum system with quadratic inertial nonlinearity are determined by the Adomian decomposition method (ADM) in Ref. [85]. Both the low energy and the high energy cases are considered, and NNMs near 1:2 and 1:1 internal resonances are analyzed. ADM also captures the possible multiple NNMs that exist in the case of internal resonance.

The new approach for calculation of the NNMs in undamped nonlinear autonomous mechanical systems is suggested in Ref. [34]. The periodic response is defined by a single one-dimensional nonlinear differential equation governing the total phase motion. Bubnov–Galerkin procedure is used to approximate NNMs in two-DOF mechanical systems. The approach based on the complex representation of the dynamic equations is suggested to obtain NNMs by Manevich [86]. Two degrees-of-freedom system is analyzed; localized NNM, in-phase and out-of-phase NNMs are treated.

Two classes of dynamical systems with energy dissipation are considered in Ref. [87].

3 Matching of NNMs Local Expansions

If NNMs modal lines are obtained as power expansions by one general coordinate, an analytical continuation of the obtained local expansions for large vibrations amplitudes can be derived using the rational diagonal Pade' approximants [14,17,88]. In same cases, it is possible to obtain NNMs for amplitudes that are changed from zero to infinity [14,17,88].

Let us consider the **n**-DOF conservative system (2.1) with potential energy $\Pi(x_1, x_2, ..., x_n)$ in the form of polynomial by $x_1, ..., x_n$. The smallest power of this polynomial is equal to 2; the largest one is equal to $2m$. The next change of the variables is used: $z_i = cx_i$, where $c = x_1(0)$; $z_1(0) = 1$. Then the potential energy can be presented in the following form:

$$V(c, z_1, ..., z_n) = \Pi[x_1(z_1), ..., x_n(z_n)] = \sum_{k=0}^{2m-2} c^k V^{(k+2)}(z_1, ..., z_n) \qquad (3.1)$$

where $V^{(r+1)}$ are homogeneous functions of $(r+1)$th degree. For small values of the amplitude **c** the linear system with potential $V^{(2)}$ can be chosen as unperturbed; the nonlinear system with homogeneous potential $V^{(2m)}$ is used for unperturbed system in the case of large amplitudes. It is assumed that the local expansions of the NNMs for small and large values of c are obtained. The local expansions for small and large amplitudes are called quasilinear and essentially nonlinear local expansions, respectively. The amplitudes of the general coordinates are presented for small and large values of c in the following power series:

$$\rho_i^{(1)} = \sum_{j=0}^{\infty} \alpha_j^{(i)} c^j; \quad \rho_i^{(2)} = \sum_{j=0}^{\infty} \beta_j^{(i)} c^{-j} \qquad (3.2)$$

In order to match the local expansions (3.2), the fractional rational diagonal two-point Pade' approximants (PA) can be used [14,17,88]. These approximants for arbitrary values of the amplitude c can be presented as

$$PA_s^{(i)} = \frac{\sum_{j=0}^{s} a_j^{(i)} c^j}{\sum_{j=0}^{s} b_j^{(i)} c^j}; \quad (s = 1, 2, ...; \, i = 2, 3, ..., n) \qquad (3.3)$$

These approximants can be rewritten with respect to c^{-1} in the following form:

$$\text{PA}_s^{(i)} = \frac{\sum\limits_{j=0}^{s} a_j^{(i)} c^{j-s}}{\sum\limits_{j=0}^{s} b_j^{(i)} c^{j-s}} \quad (s = 1, 2, \ldots; \ i = 2, 3, \ldots, n) \qquad (3.4)$$

Comparing the expressions (3.2) and (3.3, 3.4) and retaining only the terms with c^r $(-s \le r \le s)$, the system of $2(s+1)$ linear algebraic equations with respect to the coefficients $a_j^{(i)}$, $b_j^{(i)}$ is derived. The necessary condition of convergence of $\text{PA}_s^{(i)}$ to the fractional rational function PA_S at $s \to \infty$ is [14,17,88]

$$\lim_{s \to \infty} \Delta_s = 0 \qquad (3.5)$$

The composition of determinants Δ_S is considered in the paper [10].

The necessary condition for convergence of the Pade' approximants (3.5) allows to determine which of the quasilinear and essentially nonlinear local expansions correspond to the same solution and which of them to different ones. This matter is treated below in the example. Using the Pade' approximants, it is possible to compute the curvilinear NNMs for arbitrarily values of amplitudes.

Two-DOF conservative system with linear and cubic terms by the general coordinates z_1, z_2 is considered. The following change of the variables is used: $z_1 = cx$; $z_2 = cy$, where $c = z_1(0)$; $x_1(0) = 1$. Then, the potential energy can be written in the following form:

$$V = c^2 \left(d_1 \frac{x^2}{2} + d_2 \frac{y^2}{2} + d_3 xy \right)$$
$$+ c^4 \left(\gamma_1 \frac{x^4}{4} + \gamma_2 x^3 y + \gamma_3 \frac{x^2 y^2}{2} + \gamma_4 xy^3 + \gamma_5 \frac{y^4}{4} \right)$$
$$\equiv c^2 V^{(2)} + c^4 V^{(4)}$$

In the future calculations the next system parameters are used: $d_1 = 1 + \gamma$; $d_2 = 1 + \gamma$; $d_3 = -\gamma$; $\gamma_1 = 1$; $\gamma_2 = 0$; $\gamma_3 = 3$, $\gamma_4 = 0,2091$, $\gamma_5 = 2$. Then the equations of the system motions are presented in the form

$$\ddot{x} + x + \gamma(x - y) + c^2(x^3 + 3xy^2 + 0,2091y^3) = 0$$
$$\ddot{y} + y + \gamma(y - x) + c^2(2y^3 + 3x^2y + 0,6273y^2x) = 0 \qquad (3.6)$$

In this system the quasilinear local expansions for the NNMs with small values of amplitude are derived as a power series of c^2; for large values of the amplitudes of the limiting homogeneous system with cubic nonlinearity, the NNMs local expansions are obtained as a power series of c^{-2}. In both cases, the limiting homogeneous systems (linear system and essentially nonlinear one) have rectilinear NNMs $y = k_0 x$. Then it is possible to obtain the matching solutions for arbitrary values of the amplitude c using the Pade' approximants.

Now a case with two quasilinear local expansions and four essentially nonlinear local expansions is considered. Details of the calculations of local expansions for the NNMs and the Pade' approximants can be found in papers [14,17,88]. Behavior of the NNMs in the system (3.6) for different values of γ is presented in Fig. 2, where the dependence of the parameter c on $\varphi = \text{arc}tg\,[y(0)]$ is shown. As follows from this figure, two quasilinear local expansions pass to two essentially nonlinear local expansions, and vice versa; that is, these expansions correspond to two NNMs. Two others NNMs exist only in nonlinear case; when the parameter c decreases, the corresponding local expansions are joined in a saddle-node bifurcation points. If $\gamma \to 0$, the amplitude

$c \to 0$ for this limiting point. So, these additional normal modes can exist at very small amplitude values, and these NNMs take place for small values of the parameter γ. Note that the additional NNMs cannot be obtained by any quasilinear analysis.

4 Analysis of NNMS by Using Symmetries of Systems

NNMs can be associated with symmetries of a dynamical system. Then the equations of motion are invariant with respect to some group of transformations. This group-theoretical approach for computing NNMs was developed in several papers [27,89,90], where the NNMs of two-DOF systems are examined using techniques based on the theory of continuous and discrete groups.

In future consideration, the following two-DOF conservative system is considered:

$$m_i \ddot{x}_i + \sum_{k=1}^{p} \sum_{j=0}^{k} c_{kj}^{(i)} x_1^j x_2^{k-j} = 0; \quad i = 1, 2 \qquad (4.1)$$

This system can be presented in the following symbolic operator form: $S = 0$.

One considers now a continuous group of transformation, which is completely defined by its infinitesimal operator in the following form [91]:

$$U = \varepsilon(t) \frac{\partial}{\partial t} + \eta_1(x_1, x_2, t) \frac{\partial}{\partial x_1} + \eta_2(x_1, x_2, t) \frac{\partial}{\partial x_2} \qquad (4.2)$$

The condition of invariance of the system (4.1) with respect to specific group transformations may be formulated as follows:
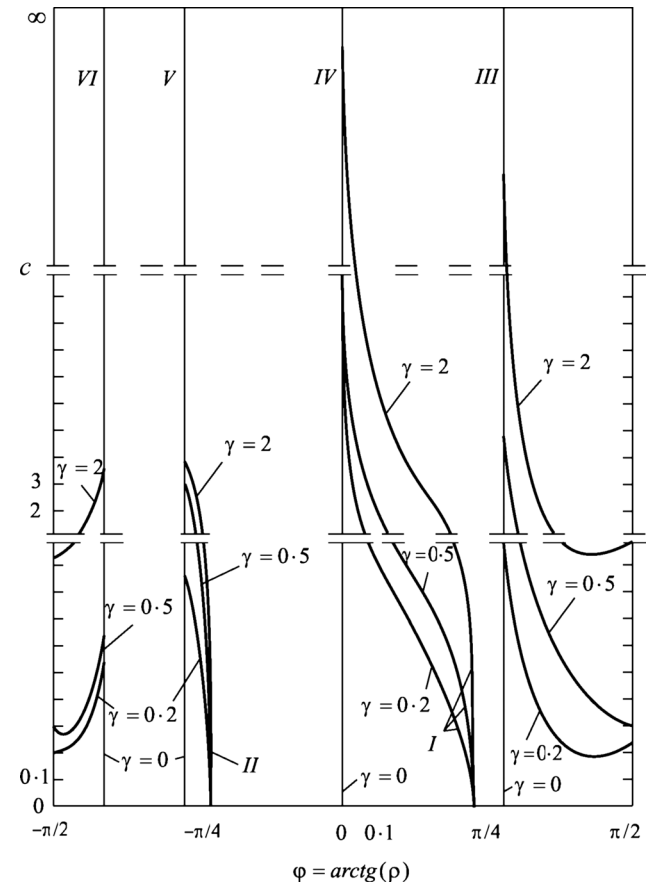

Fig. 2 Nonlinear modes of the system (3.6) for different values of parameter γ

$$U'' S_M = 0 \qquad (4.3)$$

where M is the manifold on the space $(x_1, \dot{x}_1, \ddot{x}_1, x_2, \dot{x}_2, \ddot{x}_2, t)$, which is described by the equations of motion $S = 0$; U'' is twice-continued operator [91]. After some calculation, it can be derived that the homogeneous system is invariant with respect to the group of inhomogeneous dilatations, which invariant manifolds are straight lines in a configuration space. Specific groups of transformations are used to obtain degenerate limiting systems

$$\ddot{x}_1 = 0; \quad m_2 \ddot{x}_2 + \sum_{k=1}^{p} c_{k0}^{(2)} x_2^k = 0 \qquad (4.4)$$

and

$$m_1 \ddot{x}_1 + \sum_{k=1}^{p} \sum_{j=0}^{k} c_{kj}^{(1)} x_1^j x_2^{k-j} = 0; \quad \sum_{k=1}^{p} \sum_{j=0}^{k} c_{kj}^{(2)} x_1^j x_2^{k-j} = 0 \qquad (4.5)$$

The system (4.4) has rectilinear NNM and the system (4.5) has a curvilinear one. It seems that Manevitch and Cherevatskii [92] were the first who investigated these degenerate systems.

Additional classes of limiting systems possessing the NNMs can be obtained by finding complementary discrete groups of transformations. The invariant conditions are formulated for the Lagrange functional of two-DOF conservative system. The discrete group is presented by the matrix representation S. The invariant condition for the Lagrange function $L(x, \dot{x})$ is

$$L(x, \dot{x}) = L(S^{-1}(x, \dot{x})) \qquad (4.6)$$

where $x = (x_1, x_2)^T$ is a vector of general coordinates. This condition splits into two conditions for the kinetic and potential energies. It is possible to obtain general conditions for the coefficients of the energies, which guarantee an existence of the NNMs with rectilinear modal lines [15]. Moreover, such techniques can be extended to dissipative systems and systems with gyroscopic forces. Thus, it is possible to determine new classes of mechanical systems allowing rectilinear modal lines. Moreover, assigning curvilinear modal lines, it is possible to determine the corresponding dynamical system with these modal lines. Detailed explanation of this approach is presented in papers [15,93].

Application of the group representation theory to the NNMs in symmetric multi-element continuous beam-type systems is considered in Ref. [180].

A new approach to study nonlinear dynamics of N particles physical systems with discrete symmetries was developed in Refs. [28, 29]. This method is based on the concept of bushes of normal modes that represent a certain class of exact nonlinear solutions. This class includes both one-dimensional similar Kauderer–Rosenberg NNMs and these mode interactions. Transfer between modes with different symmetry causes the existence of mode bushes [28,29]. Each bush possesses its own symmetry group which is a subgroup G_j of the parent group G_0 (the group of dynamical system invariance). Such subgroups can be determined at the arbitrary instant of time. The technique for irreducible representations of the symmetry groups is used for the bushes analysis. The N particle physical systems are characterized by 230 space groups of a crystallographic symmetry. It was found that for such systems with analytical potentials, 19 classes of irreducible bushes of nonlinear modes exist. For each of these symmetrical systems, the potential energy was obtained as a power series of general coordinates up to sixth degree. All possible flows with quadratic nonlinearities, which are invariant under the action of 32 point groups of crystallographic symmetry, were obtained.

The displacements of all the chain particles are described by time periodic functions with the same frequency in the paper [29]. Discrete breathers in monoatomic chains were considered as generalization of the Hamiltonian chains NNMs. This analysis was used to study NNMs, which are determined by the group-theoretical approach in the Fermi-Past-Ulam chains [94] and to study three-dimensional flows with quadratic nonlinearities [95].

5 Stability and Bifurcations of NNMs

5.1 Stability of the Rectilinear NNMs. One considers a problem of the rectilinear NNMs stability. The following procedure of the stability analysis can be suggested for n-DOF systems. The new variables $(\xi, \eta_1, ..., \eta_{n-1})$ are introduced. The ξ-axis is directed along the rectilinear trajectory and $\eta_1, ..., \eta_{n-1}$ axes have the orthogonal directions. Stability along the rectilinear trajectory is not considered. The small orthogonal perturbations $\Delta\eta_1, ..., \Delta\eta_{n-1}$ define a stability of the trajectory in the system configuration space. The stability problem after linearization on $\Delta\eta_1, ..., \Delta\eta_{n-1}$ is reduced to analysis of linked linear differential equations. A pattern of the small perturbations behavior in configuration space, close to the unstable rectilinear NNM of two-DOF mechanical system, is shown in Fig. 3.

In first publications of Rosenberg and his co-authors [4,67] on the stability of rectilinear NNMs in two-DOF systems, the harmonic approximation of the periodic motions was used. The well-known Mathieu equation was obtained. Rand [96] obtained linear ordinary differential equation, which describes orthogonal motions close to the curvilinear NNMs of two-DOF system. This equation includes a curvature of the modal line as a system parameter. Pecelli and Tomas [97,98] considered variational equations for rectilinear NNMs in the form of the Lame equations. Boundaries of the instability regions are treated. Moreover, the authors suggested the method for the NNMs stability analysis using nonlinear variational equations. The Arnold−Moser theory is used in this analysis.

The NNMs stability analysis is performed by Ince algebraization method, which is treated in the book [99]. The algebraization is carried out by choosing a new independent variable x associated with the NNM instead of the variable t. The new independent variable defines the motions along the rectilinear NNM. Then the variational equations with periodic coefficients are transformed into equations with singular points.

The separation of the variational equations for rectilinear modal lines is considered in [100,101]. It is shown that the stability problem is decomposed into four Sturm–Liouville problems for even and odd solutions of the variational equations.

5.2 Stability of NNMs in Homogeneous Systems. The mechanical systems with the potential energy

$$\Pi = \sum_{\alpha_1 + \alpha_2 + ... + \alpha_n = r} a_{\alpha_1 \alpha_2 ... \alpha_n} (r+1)^{-1} x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$$

which is even homogeneous function with respect to the general coordinates of power $r + 1$ are considered. Such systems have
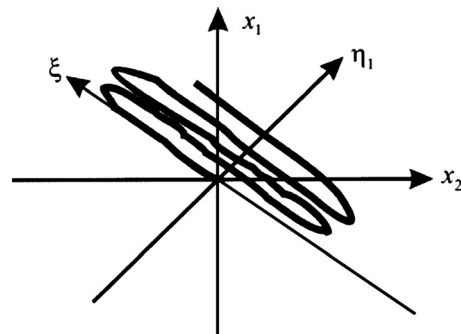

Fig. 3 The behavior of small perturbations close to modal line $\eta_1 = 0$ in configuration space of two-DOF system

rectilinear NNMs $x_i = k_i x_1$. The motions along these NNMs are described by the following equation: $\ddot{x} + \Omega^2 x^r = 0$. The vector of perturbations $V = (v_1, v_2, ..., v_{n-1})^T$ in the orthogonal directions to the NNM is used. The variational equations with respect to this vector are the following: $\ddot{V} + A_r x^{r-1} V = 0$. The symmetric matrix A_r can be diagonalized by using the linear nondegenerate orthogonal transformation, and the system of variational equations can be made uncoupled to separate equations. All separate equations are presented in the following form: $d^2 v/dt^2 + q x^{r+1} v = 0$; $(q = const)$. The new independent variable is used: $z = (\Omega^2 x^{r+1})/(h(r+1))$. As a result, the variational equation is presented in the following form:

$$z(z-1)v_{zz} + \left(\frac{r}{r-1} - \frac{3r+1}{2(r+1)}z \right)v_z + \lambda v = 0 \qquad (5.1)$$

where $\lambda = q/2(r+1)\Omega^2$; $v_z = dv/dz$. Equation (5.1) is hypergeometric with singular points $z = 0$; $z = 1$. Solutions, corresponding to the boundaries of stable motions in the system parameter space, are called degenerate solutions of the hypergeometric equation. The values of λ, corresponding to the "boundary" solutions, are called eigenvalues of the stability problem. These "boundary" solutions can be presented as $v = z^{\mu_1}(1-z)^{\mu_2}q_n(z)$, where $q_n(z)$ are polynomials; the parameters μ_1, μ_2 are equal to 0, $1/(r+1)$ and 0, $1/2$, respectively. The degenerate solutions are orthogonal Gegenbauer polynomials [102]. Note that the solutions are periodic in time with period equal to T or $T/2$, where T is a period of the motion $x(t)$. The eigenvalues of the stability problem are the following:

$$\lambda_{4j} = j[(2j+1)(r+1) - 2]; \quad \lambda_{4j+1} = (2j+1)[j(r+1)+1]$$
$$\lambda_{4j+2} = (2j+1)[(j+1)(r+1) - 1] \qquad (5.2)$$
$$\lambda_{4j+3} = (j+1)[(2j+1)(r+1)+2] \, (j = 0, 1, 2, ...)$$

Now the stability of the NNMs of the following two-degree-of-freedom mechanical system is considered:

$$m_1 \ddot{x}_1 + a_{11} x_1^r + a_{12}(x_1 - x_2)^r = 0$$
$$m_2 \ddot{x}_2 + a_{22} x_2^r + a_{12}(x_2 - x_1)^r = 0 \qquad (5.3)$$

where r is odd positive number. The system (5.3) has the rectilinear NNMs $x_2 = k x_1$, where k is determined from the following equation:

$$(1-k)^r (1 + \kappa_3 k)\kappa_2 = \kappa_1 k^r - \kappa_3 k \left(\kappa_1 = \frac{a_{22}}{a_{11}}, \kappa_2 = \frac{a_{12}}{a_{11}}, \kappa_3 = \frac{m_2}{m_1} \right)$$
$$(5.4)$$

The eigenvalues of the system (5.3) stability problem are the following [14,100]:

$$\lambda = \frac{r(1 + \kappa_3 k)(\kappa_1 k^{r-1} - \kappa_3 k)}{\kappa_3 (1-k)(\kappa_1 k^r + 1)} \qquad (5.5)$$

As follows from the analysis of the Eqs. (5.4) and (5.5) [100,101], the in-phase NNM exists for positive values of the parameter κ_2. Moreover, depending on the parameter r, the system (5.3) has single, three or five out-of-phase NNMs. Figure 4 shows the dependence of the NNMs parameter $\xi = \arctan k$ on the value of $\eta = 4\pi^{-1} \arctan(\kappa_1 2^r/(r-1))$. Curves labeled by a, b, c, and d correspond to the following system parameters: $(r, \kappa_1) = (3, 1)$; $(r, \kappa_1) = (7, 1)$; $(r, \kappa_1) = (3, 0)$; $(r, \kappa_1) = (3, 1.2)$. The stability of the NNMs is changed in the points, which are marked by "$\bigcirc$" for $r = 3$ and by "$\times$" for $r = 7$. Details of this analysis can be found in Refs. [100,101].
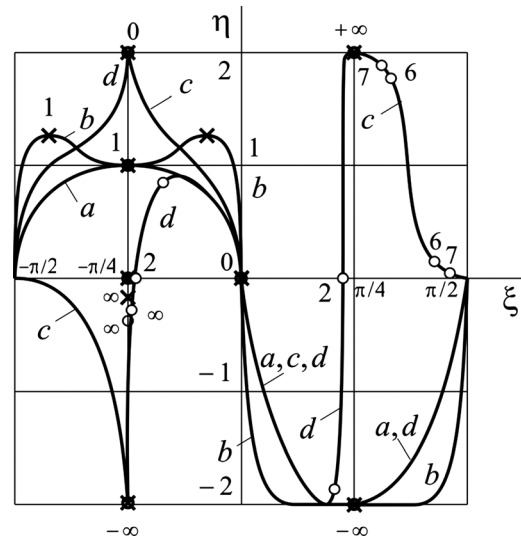

Fig. 4   Dependence of the NNMs on parameter η

Bifurcations of the NNMs are also considered. Classification of the forking motions in homogeneous systems is made in Refs. [100,101]. The representatives of the forking trajectories in a configurational space are qualitatively shown in Fig. 5. The forking trajectories similar to the rectilinear and parabola are shown in Figs. 5(a) and 5(d). If the "parabola" pass through the equilibrium $(x, y) = (0, 0)$, the single forking trajectory exists, but if such "parabola" does not pass through this equilibrium, the couple of symmetric forking trajectories appear. It is obvious that such forking motions belong to the Kauderer–Rosenberg NNMs. The forking trajectories similar to ellipse or eight curve in a configuration space are shown in Figs. 5(b) and 5(c). Details of this classification are described in Refs. [100,101].

More general results on the stability and bifurcations of NNMs for the systems close to the homogeneous are obtained by using the perturbation theory [14,15,60].

5.3   Finite Zoning Conditions for the Stability Problem. In general case, a number of instability regions of periodical motions is infinite. As an example, it is possible to mention the well-known Ince–Strutt diagram, which is described the domains of instabilities in Mathieu equation. Here it is considered a case when a number of instability zones for the NNMs is finite and the other instability zones shrink to lines. This case takes place if the system parameters satisfy specific conditions [14,101,103]. If these conditions are not satisfied, a number of instability zones is infinite, but if the values of the system parameters are close to the values that guarantee a realization of the finite zoning conditions, then all instability zones, except a finite number of them, are very narrow.

The following two-DOF conservative system is considered: $\ddot{x}_i + \partial\Pi/\partial x_i = 0$; $(i = 1, 2)$. It is assumed that the system allows rectilinear NNMs. The coordinate axes are rotated; the general coordinates (x_1, x_2) are transformed into $(\tilde{x}_1, \tilde{x}_2)$ so that the transformed system has the NNM $\tilde{x}_2 = 0$. The system potential energy can be presented as
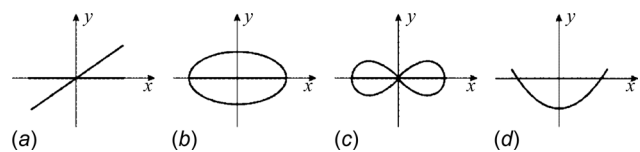

Fig. 5   The forking trajectories near the rectilinear NNM

$$\Pi = \sum_{i=2}^{m} a_i \tilde{x}_1^i + \tilde{x}_2^2 \sum_{i=0}^{m-2} e_i \tilde{x}_1^i + \sum_{i=3}^{\infty} \tilde{x}_2^i g_i(\tilde{x}_1^i) \qquad (5.6)$$

where a_i, e_i are constant. The condition of existence of the NNM $\tilde{x}_2 = 0$ is the following: $\partial \Pi / \partial \tilde{x}_2 (\tilde{x}_1, 0) = 0$. In order to study the NNM stability, only the perturbations y orthogonal to NNM modal line are considered. Then the variational equation is

$$\ddot{y} + y \, \Pi_{x_2 x_2}(\tilde{x}_1, 0) = 0. \qquad (5.7)$$

The finite zoning conditions can be obtained by two different ways. One approach is proposed for the equation with a periodical coefficient $\ddot{\psi} + [\lambda - u(t)] \psi = 0$ by Novikov and his co-authors [104,105]. Application of this criterion to three classes of the nonlinear conservative systems with rectilinear NNMs is considered in Refs. [14,101,103]. Other approach is used the Ince-algebraization. An algebraic form of the variational Eq. (5.7) can be obtained by using the new independent variable x instead of t. This transformation is similar to one, which was treated in Sec. 5.1. The Eq. (5.7) can be presented in the following algebraic form:

$$2[h - \Pi(x,0)]y_{xx} - \Pi_x(x,0)y_x + \Pi_{x_2 x_2}(x,0)y = 0 \qquad (5.8)$$

Singularities of this equation are zeros of the function $h - \Pi(x,0)$. The eigenfunctions, corresponding to the boundaries of stability/instability regions, can be presented as

$$y = p(x) \, ; \quad y = \sqrt{x - \xi} R(x) \qquad (5.9)$$

where $p(x)$, $R(x)$ are analytical functions; $x = \xi$ is a singularity of the Eq. (5.8).

The following three classes of the potential energy (5.6) are considered:

   (I) $a_2 \neq 0, \, a_4 \neq 0, \, e_0 \neq 0, \, e_2 \neq 0$
   (II) $a_2 \neq 0, \, a_4 \neq 0, \, a_6 \neq 0, \, e_0 \neq 0, \, e_2 \neq 0, \, e_4 \neq 0$
   (III) $a_2 \neq 0, \, a_3 \neq 0, \, a_4 \neq 0, \, e_0 \neq 0, \, e_1 \neq 0, \, e_2 \neq 0$

All others coefficients of the potential energy (5.6) are equal to zero. First of all, the case (I) is considered. Then the variational Eq. (5.8) coincides with the following Lame equation:

$$y''(z^2 - a^2)(z^2 - b^2) + y'z(2z^2 - a^2 - b^2) - y[n(n+1)z^2 - \lambda] = 0 \qquad (5.10)$$

if $z \equiv x$; $h/a_4 = -a^2 b^2$; $a_2/a_4 = -a^2 - b^2$; $e_0/a_4 = -\lambda$; $e_2/a_4 = n(n+1)$. The condition of finite zoning is the following: $e_2/a_4 = n(n+1), \, n = 0,1,2,\dots$. In the case (II), the variational equation is reduced to the Lame equation (5.10) by the following transformation of the variable $z = x^2 + 0.25\,(a_4/a_6)$, if the following relations are true:

$$\frac{h}{a_6} = 2a(a^2 - b^2); \qquad \frac{a_2}{a_6} = 5a^2 - b^2; \qquad \frac{a_4}{a_6} = -4a;$$
$$\frac{e_0}{a_6} = 8\left(a^2 n(n+1) - \lambda\right).$$

In this case, the finite zoning condition is the following: $e_4/a_6 = 4n(n+1), \, n = 0,1,2,\dots$. In the case (III), the variational equation is reduced to the Lame equation by using the coordinate transformation $z = x + \mu$ if the following relations are true:

$$\frac{h}{a_4} = \frac{1}{4}(a^2 - b^2)^2; \qquad \frac{a_2}{a_4} = 2(a^2 + b^2); \qquad \frac{a_3}{a_4} = 4\mu;$$
$$\frac{e_0}{a_4} = -\lambda + \mu^2 n(n+1); \qquad \frac{e_1}{a_4} = 2\mu n(n+1),$$

where $\mu = \pm \sqrt{(a^2 + b^2)/2}$. The finite zoning condition is the following: $e_2/a_4 = n(n+1), \, n = 0,1,2,\dots$.
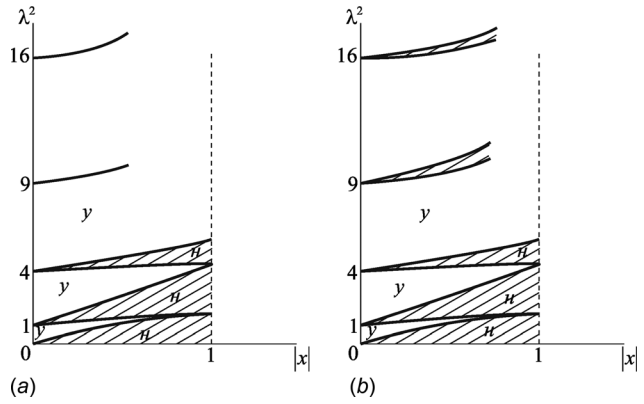
Fig. 6 Plane of stability of the Lame equation solutions: (a) finite zoning case; (b) infinite zoning case

Figure 6 shows two maps of stability of the Lame equation solutions. Figure 6(a) presents three instability regions when all other instability regions are collapsed to lines; Fig. 6(b) shows the infinite numbers of the instability regions.

5.4 Bifurcations of the NNMs With Curvilinear Modal Lines. Bifurcations of the curvilinear NNMs, which are close to the rectilinear ones, are analyzed using the perturbation approach in Ref. [14]. Two different types of periodic solutions of variational equations, which affect to the NNMs stability, are considered. These periodic solutions can be presented in the following form:

$$y = \alpha(x_0); \text{ or } \sigma_2 = \dot{x}_0 \beta(x_0)$$

where $\alpha(x_0)$ and $\beta(x_0)$ are analytical functions; $x_0 = x_0(t)$ is a periodic motions, corresponding to the NNM. Trajectories of the bifurcation solutions are similar to ones, which are shown in Fig. 5. There are parabolic-type trajectories with few half-waves and trajectories of the elliptic type with few loops.

The paper [106] presents an analysis of the NNMs stability and bifurcations. The characteristic exponents for the associated Hill's equation are estimated. Using the local bifurcation analysis techniques, Rand et al. [107] and Pak et al. [108] analyzed bifurcations of NNMs and periodic motions in two-DOF conservative systems exhibiting 1:1 resonance. The harmonic approximation is consistent with the perturbation technique, which is used to study bifurcations of the NNMs in Ref. [107]. Vakakis and Rand [16,109] investigated the NNMs of the conservative two-DOF systems. They studied an appearance of additional normal modes via a pitchfork bifurcation. It is showed that the bifurcation of similar normal modes results in chaotic motions, which do not exist in the system before this bifurcation. An effect of slightly variation of the parameters on the regions of NNM instability is analyzed in Ref. [110].

Uncoupled and coupled NNMs (so-called elliptic modes) for free and forced vibrations are analyzed in a neighborhood of the 1:1 resonance in nonlinear two-DOF system with cubic nonlinearity by the multiple-scale method [111–114]. The mechanical system with cubic nonlinearity and two internal resonances is considered in Ref. [115]. The bifurcation analysis of NNMs is performed on the basis of modulation equations obtained by the multiple-scale method. NNMs of two-DOF autonomous system are considered in the paper [84]. The multiple-scale method and calculations of Poincare' sections are used to study bifurcations of NNMs.

The coupled modes are considered in Ref. [83]. These motions are called bifurcation modes as they appear in points where the NNM lose stability. Two new NNMs appear from nowhere in a generic bifurcation; an existing NNM throws off two new NNMs

and continues to exist but lost a stability in a nongeneric bifurcation. The generic bifurcation is saddle-node, and the nongeneric bifurcation (super- or subcritical) is pitchfork one. The NNMs in a case of internal resonances and bifurcated modes are considered in the paper [116]. The Synge's concept of stability is used in Ref. [117]. In order to analyze the NNM bifurcations, the harmonic balance method is used. Super- and subcritical nongeneric bifurcations are analyzed. The properties of NNMs are connected with global properties of the Poincare' section.

5.5 Stability of NNMs in High Approximations (Nonlinear Stability). Nonlinear stability of NNMs can be studied after an analysis of the linear problem. The nonlinear stability was considered for the two-DOF conservative systems in Refs. [97,98] using the Siegel and Moser approach [118]. The analysis is made for the system having restoring forces with linear and cubic terms with respect to general coordinates. First of all, the system is transformed to the single-DOF nonautonomous system by using the energy integral [119,120]. Solutions of the system are presented as power series by initial values of variables. The first coefficients of these series are determined from the linearized variational equations. Then the Poincare' map is constructed and the Arnold–Moser–Russman theorem is used. It is obtained the instability regions in the system parameter space, which are derived from the nonlinear analysis. These regions have smaller dimension than the instability regions derived from the linearized variational equations.

The method for determining the NNMs stability in autonomous two degrees-of-freedom Hamiltonian system is suggested by Month and Rand [121]. They show that the linear stability analysis sometimes fails to predict stability of in-phase mode. The alternative approach, which does not require linearization in the neighborhood of any particular motions, is suggested. The basis of this approach is the Poincare'-Birkhoff nonlinear forms.

6 Shaw–Pierre NNMs

Shaw and Pierre [122,123] reformulated the concept of NNMs for a general class of nonlinear finite-DOF mechanical systems with dissipation. The analysis is based on the center manifolds theory, which is presented in Refs. [37,124]. The Shaw–Pierre NNMs has several essential differences compared to the Kauderer–Rosenberg NNMs. In the first place, the energy integral does not need for Shaw–Pierre NNMs construction, even in conservative systems. Thus, the Shaw–Pierre approach allows an effective analysis of systems with dissipation. In the second place, this approach is easily used to analyze the mechanical systems with many degrees-of-freedom. In the third place, it is possible to construct both periodic motions and transient on manifolds. Therefore, Shaw–Pierre NNMs are widely used to analyze the vibrations of elastic systems, which are considered in the next review paper by the same authors. A comparison of the center manifolds and the Shaw–Pierre NNMs are made in Ref. [125].

The NNM invariant manifolds of the N-DOF nonlinear system without internal resonance are two-dimensional; the invariant manifolds of the N-DOF nonlinear system with internal resonances have extended dimension due to a nonlinear interaction between NNMs that couple them.

Now the main ideas of Shaw–Pierre NNMs are treated. Autonomous N-DOF mechanical systems with stable equilibrium at origin are considered. This system can be presented as

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = f(x, y) \qquad (6.1)$$

where $x = \{x_1 \ldots x_N\}^T$; $y = \{y_1 \ldots y_N\}^T$ are vectors of the generalized coordinates and velocities; $f = \{f_1 \ldots f_N\}^T$ is a vector of the forces. The system (6.1) can be presented in the following form:

$$\ddot{x}_i + \beta_i \dot{x}_i + \omega_i^2 x_i = \tilde{f}_i(x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N) \quad i = 1, \ldots, N \qquad (6.2)$$

where β_i are coefficients of linear damping; $\tilde{f}_i$ are nonlinear forces acting on mechanical system. Note that a case of internal resonances in the system (6.1) is not considered. Variables of the system (6.1) are divided into the following two groups: the master coordinates and the slave coordinates. The master coordinates are independent variables of the NNMs. For example, the following variables can be used as a master coordinates: $u \equiv x_1$; $v \equiv y_1$. The rest variables of the system (6.1) are named as slave coordinates. Following the Shaw–Pierre approach, all slave coordinates are single-valued function of the master coordinates. Thus, the nonlinear mode can be presented as

$$\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_N \\ y_N \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ X_2(u,v) \\ Y_2(u,v) \\ \vdots \\ X_N(u,v) \\ X_N(u,v) \end{Bmatrix} \qquad (6.3)$$

Computing the derivatives of (6.3) and using the system (6.1), the following system of the partial derivation equations is derived:

$$\frac{\partial X_i(u,v)}{\partial u}v + \frac{\partial X_i(u,v)}{\partial v}f_1(u,v,X_2,Y_2,\ldots,X_N,Y_N) = Y_i(u,v)$$
$$\frac{\partial Y_i(u,v)}{\partial u}v + \frac{\partial Y(u,v)}{\partial v}f_1(u,v,X_2,Y_2,\ldots,X_N,Y_N)$$
$$= f_i(u,v,X_2,Y_2,\ldots,X_N,Y_N); \quad i = 2,\ldots,N \qquad (6.4)$$

Following the book [37], the solution of the system (6.4) is presented in the form of power series:

$$x_i = X_i(u,v) = a_{1i}u + a_{2i}v + a_{3i}u^2 + a_{4i}uv + a_{5i}v^2 + \ldots$$
$$y_i = Y_i(u,v) = b_{1i}u + b_{2i}v + b_{3i}u^2 + b_{4i}uv + b_{5i}v^2 + \ldots \qquad (6.5)$$
$$i = 2,\ldots,N$$

where a_{1i}; a_{2i}; $\ldots$; b_{1i}; b_{2i} are unknown parameters. Such series was used by Lyapunov [1] to construct periodical solutions in quasilinear systems. The expansions (6.5) are substituted into (6.4); the coefficients of the same powers on $u^{l_1}v^{l_2}$; $l_1 = 0, 1, \ldots$; $l_2 = 0, 1, \ldots$ are equated. The coefficients a_{1i}; a_{2i}; b_{1i}; b_{2i}; $i = 2, \ldots, N$ satisfy a system of nonlinear algebraic equations, and the coefficients a_{3i}; a_{4i}; $\ldots$; b_{3i}; b_{4i}; $\ldots$ satisfy a system of linear algebraic equations. These systems are treated in Ref. [123]. Thus, the coefficients of the expansions (6.4) are determined from these equations; as a result, the NNM is determined. Note that the coefficients a_{1i}; a_{2i}; b_{1i}; b_{2i}; $i = 2, \ldots, N$ can be obtained without an analysis of the system of nonlinear algebraic equations. The corresponding method for simple determination of these coefficients is suggested in Refs. [50,126].

By using the above-presented approach, the invariant manifolds are calculated in the form of power series. Domain of validity of the resulting solution is limited to some neighborhood of the system equilibrium position. The Bubnov–Galerkin based approach is considered in Ref. [19]; it permits to construct NNMs, which are accurate in significantly higher region close to equilibrium position in comparison with the invariant manifolds of the form (6.5). Now this approach is treated; the dynamical system in the form (6.2) is considered. The variables $x_k, \dot{x}_k$ are chosen as master coordinates and all others phase coordinates of the system (6.2) are slaves. The NNMs of the system (6.2) can be presented as $x_i = X_i(x_k, \dot{x}_k)$; $\dot{x}_i = Y_i(x_k, \dot{x}_k)$; $i = 1, \ldots, k-1, k+1, \ldots, N$. The following change of the master coordinates is used: x_k

$= a \cos\phi$; $\dot{x}_k = -a\omega_k \sin\phi$. Thus, the NNM can be presented in the following form:

$$x_i = P_i(a, \phi) ; \quad \dot{x}_i = Q_i(a, \phi) ; \quad i = 1, ..., k-1, k+1, ..., N \tag{6.6}$$

The functions of invariant manifolds P_i, Q_i are periodic in the variable ϕ and they can be presented as Fourier series. The motions of the system (6.2) on the invariant manifold can be described by the system of differential equations with respect to (a, ϕ)

$$\dot{a} = -\sin\phi \left(\frac{\tilde{f}_k}{\omega_k} + \beta_k a \sin\phi \right)$$
$$\dot{\phi} = \omega_k - \cos\phi \left(\frac{\tilde{f}_k}{\omega_k a} + \beta_k \sin\phi \right) \tag{6.7}$$

The functions P_i, Q_i are described by the following system of partial differential equations:

$$-\frac{\partial P_i}{\partial a} \sin\phi \left(\frac{\tilde{f}_k}{\omega_k} + \beta_k a \sin\phi \right) + \frac{\partial P_i}{\partial \phi} \left\{ \omega_k - \cos\phi \left(\frac{\tilde{f}_k}{\omega_k a} + \beta_k \sin\phi \right) \right\}$$
$$= Q_i$$
$$-\frac{\partial Q_i}{\partial a} \sin\phi \left(\frac{\tilde{f}_k}{\omega_k} + \beta_k a \sin\phi \right) + \frac{\partial Q_i}{\partial \phi} \left\{ \omega_k - \cos\phi \left(\frac{\tilde{f}_k}{\omega_k a} + \beta_k \sin\phi \right) \right\}$$
$$= -\beta_i Q_i - \omega_i^2 P_i + \tilde{f}_i; \quad i = 1, ..., k-1, k+1, ..., N \tag{6.8}$$

The partial differential equations (6.8) are solved by using the Bubnov–Galerkin based approach, which is treated in Ref. [19].

Let us consider the computational costs to determine NNM (6.5) and (6.6). The system of linear algebraic equations with respect to coefficients of the expansion (6.4) is solved to determine the nonlinear mode (6.5), and the system of nonlinear algebraic equations is calculated to determine the nonlinear mode (6.6). Thus, the numerical determination of NNM (6.6) is a more complex problem in comparison with the calculation of the NNM (6.4).

Multimode invariant manifolds in phase space, which is a generalization of the nonlinear modes (6.5), (6.6), is used to analyze the system dynamics with internal resonances. Such invariant manifolds are analyzed in Refs. [20,21]. It is assumed that the eigenfrequencies of M modes of linear vibrations are involved in an internal resonance. These modes are described by a set of indexes, which are denoted by S_M. According to the definition of invariant manifolds, phase variables involved in the internal resonance $(u_k, v_k) = (x_k, \dot{x}_k)$; $k \in S_M$ are chosen as master coordinates. Then all others phase variables are slave coordinates. $(2N - 2M)$ slave coordinates are determined through the nonlinear mode in the following way:

$$x_j = X_j(u_m; v_m)$$
$$y_j = Y_j(u_m; v_m); \quad j \notin S_M; \quad m \in S_M \tag{6.9}$$

The partial differential equations, describing the invariant manifold, have the following form [20,127]:

$$\sum_{k \in S_M} \left[\frac{\partial X_j}{\partial u_k} v_k + \frac{\partial X_j}{\partial v_k} f_k \right] = Y_j$$
$$\sum_{k \in S_M} \left[\frac{\partial Y_j}{\partial u_k} v_k + \frac{\partial Y_j}{\partial v_k} f_k \right] = f_j ; \quad j \notin S_M \tag{6.10}$$

Local approximation of the manifold can be found by means of the Taylor series

$$X_j(u_m, v_m) = \sum_{k \in S_M} \left[a_{1,j}^{(k)} u_k + a_{2,j}^{(k)} v_k \right] + \sum_{k \in S_M} \sum_{l \in S_M}$$
$$\times \left[a_{3,j}^{(k,l)} u_k u_l + a_{4,j}^{(k,l)} u_k v_l + a_{5,j}^{(k,l)} v_k v_l \right] + ...$$
$$Y_j(u_m, v_m) = \sum_{k \in S_M} \left[b_{1,j}^{(k)} u_k + b_{2,j}^{(k)} v_k \right] + \sum_{k \in S_M} \sum_{l \in S_M}$$
$$\times \left[b_{3,j}^{(k,l)} u_k u_l + b_{4,j}^{(k,l)} u_k v_l + b_{5,j}^{(k,l)} v_k v_l \right] + ... \tag{6.11}$$

The expansions (6.11) are substituted into (6.10); the coefficients of the summands with $u_k^{j_1} v_l^{j_2}$ are collected and equated. Then the coefficients $a_{1,j}^{(k)}$; $a_{2,j}^{(k)}$ satisfy the system of nonlinear algebraic equations and coefficients $a_{3,j}^{(k,l)}$; $a_{4,j}^{(k,l)}$;... are determined from the system of linear algebraic equations.

If a system of linear algebraic equations with respect to the coefficients $a_{3,j}^{(k,l)}$; $a_{4,j}^{(k,l)}$;... has a determinant close to zero, it means that the master coordinates are chosen improper. So, other master coordinates must be chosen, and the nonlinear mode (6.11) will be calculated once more.

The application of the Bubnov–Galerkin method for the multimode invariant manifold analysis of the system (6.2) without damping $(\beta_i = 0)$ is considered in Ref. [21]. It is assumed that the functions $\tilde{f}_i$ do not depend on the general velocities. Each pair of state variables involved in the internal resonances $(x_k, \dot{x}_k)$; $k \in S_M$ are chosen as master coordinates. Then the multimode invariant manifold can be presented in the form (6.9). The polar coordinate transformation is applied for each pair of the master coordinates $x_k, \dot{x}_k$

$$(x_k, \dot{x}_k) = a_k(\cos(\phi_k), -\omega_k \sin(\phi_k)) ; \quad k \in S_M \tag{6.12}$$

where (a_k, ϕ_k) are polar master coordinates. All slave coordinates are expressed as functions of polar master coordinates in the following form:

$$q_i = P_i(a_k, \phi_k)$$
$$\dot{q}_i = Q_i(a_k, \phi_k) ; \quad k \in S_M; i \notin S_M \tag{6.13}$$

The equations of the system motions on the multimode invariant manifold can be presented with respect to the polar coordinates as

$$\dot{a}_k = -\frac{\tilde{f}_k}{\omega_k} \sin\phi_k$$
$$\dot{\phi}_k = \omega_k - \frac{\tilde{f}_k}{\omega_k a_k} \cos\phi_k; \quad k \in S_M \tag{6.14}$$

The partial differential equations of the NNM in the form (6.13) are derived in Ref. [21].

The Shaw–Pierre nonlinear modes are used for analysis of self-vibrations. Applying this approach, it is possible to study vibrations close to the Andronov–Hopf bifurcations [50].

NNMs of assembly of nonlinear component structures are considered in the paper [128]. The NNMs are constructed for the various substructures using the invariant manifold approach and then coupled through the traditional linear modes.

The theory of normal forms of ordinary differential equations is considered jointly with NNMs in the papers [129,130]. It is shown that NNMs can be calculated by using these normal forms. NNMs in the case of internal resonances are analyzed in the paper [131]. The multiple-scale method is used to obtain NNMs. The Shaw–Pierre NNMs and homoclinic orbits in two-DOF nonlinear systems are discussed in Ref. [132].

Touze and Amabili [133] considered finite-degree-of-freedom mechanical systems with quadratic and cubic nonlinearities and linear viscous damping. The method of normal forms is applied to analyze the mechanical system. As a result, the simplified

dynamical system is derived. NNMs are analyzed in the obtained simplified system.

7 NNMs in Nonautonomous Systems

Periodic solutions in nonautonomous systems close to the Lyapunov systems are investigated by Malkin [22]. It seems that Rosenberg [23,134] was the first who used the Rauscher method to analyze NNMs of nonautonomous systems. It is assumed that the unperturbed autonomous system is homogeneous. NNMs in finite-DOF nonautonomous mechanical systems close to conservative ones with similar NNMs were considered in Ref. [24]. Generalization of the Rauscher's approach and the power-series method to finite-DOF systems are used jointly to obtained resonance periodic motions. The development of the theory of NNMs in nonautonomous systems is considered in Refs. [26,135–137]. NNMs in near-conservative self-excited systems were analyzed in Ref. [25]. NNMs of parametric vibrations are treated in Ref. [26]. The NNMs (so-called resonance modes) are investigated in two coupled self-excited and parametrically excited oscillators [138].

7.1 Generalization of Rauscher Method.
The perturbation methodology is applied to wide classes of finite-DOF nonautonomous and self-excited systems close to conservative ones. The following nonautonomous dynamical system is considered:

$$\ddot{x}_i + \Pi_{x_i}(x_1, x_2, ..., x_n) + \varepsilon f_i(x_1, \dot{x}_1, x_2, \dot{x}_2, ..., x_n, \dot{x}_n, t) = 0$$
$$i = 1, ..., n \tag{7.1}$$

where f_i can describe a friction or interaction of the system with gas or fluid.

The periodic motions when all phase coordinates are analytical functions of one general coordinate $x \equiv x_1$ are considered. Besides, it is assumed that within a semi period of the motions, the variable t is a single-valued function of the displacement x. This idea was suggested by Rauscher [139] for the single-DOF nonlinear nonautonomous system. Thus, the solutions of the system (7.1) can be presented as

$$x_i = x_i(x, \varepsilon); \ \dot{x} = \dot{x}(x, \varepsilon); \ \dot{x}_i = \dot{x}_i(x, \varepsilon); \ t = t(x, \varepsilon) \tag{7.2}$$

Along the modal line the nonconservative system behaves like a single-DOF pseudo-autonomous system [14,15,24]. Such periodic solutions could be called NNMs of the nonconservative nonlinear systems.

Introducing a new independent variable x and eliminating time t using the Eq. (2.5), the following equations governing the NNMs trajectories are derived [14,15,24]:

$$2x_i'' \dot{x}^2(x) + x_i'[-\Pi_x(x, x_2(x), ..., x_n(x)) + \varepsilon f_1(x, \dot{x}(x), x_2(x), ..., t(x))]$$
$$+ \Pi_{x_i}(x, x_2(x), ..., x_n(x)) + \varepsilon f_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x), ..., t(x)) = 0$$
$$i = 2, ..., n \tag{7.3}$$

The Eq. (7.3) has singularities in the modal lines return points, where all velocities are equal to zero. Similarly to conservative systems, these singularities are removed by using the following boundary conditions:

$$\{-\Pi_x(x, x_2(x), ..., x_n(x))x_i' + \varepsilon f_1(x, \dot{x}(x), x_2(x), \dot{x}_2(x), ..., t(x)) \ x_i'$$
$$+ \Pi_{x_i}(x, x_2(x), ...) + \varepsilon f_i(x, \dot{x}(x), x_2(x), \dot{x}_2(x), ..., t(x))\}|_{x=X_j} = 0$$
$$i = 2, ..., n; \ j = 1, 2 \tag{7.4}$$

The Eq. (7.3) and boundary conditions (7.4) are similar to (2.7) and (2.8), which are used to calculate the NNMs of the conservative systems. Thus, the single-valued modal lines $x_i(x)$ close to the rectilinear NNM $x_{i0} = k_i x$ ($k_i = const$) are obtained. Moreover, the modal lines can be determined as a series of ε and x by

the method, which is treated in Sec. 2.3. If modal line in configuration space is found, the equation of motion along this modal line is the following:

$$\ddot{x} + \Pi_x(x, x_2(x), ..., x_n(x)) + \varepsilon f_1(x, \dot{x}(x), x_2(x), \dot{x}_2(x), ..., t(x)) = 0 \tag{7.5}$$

Thus, along this NNM, the nonautonomous system behaves like a single-DOF mechanical system, whose motion is determined by the general coordinate $x(t)$. The solution of the system (7.5) can be presented in the following form:

$$t(x, \varepsilon) = \frac{1}{\sqrt{2}} \int_{x_0}^{x} \frac{d\xi}{\sqrt{h(\varepsilon) - \Pi(\xi, x_2(\xi), ..., x_n(\xi)) - \varepsilon F(\xi, \varepsilon)}} - \varphi \tag{7.6}$$

where $F(x, \varepsilon) = \int f_1(x, \dot{x}(x), x_2(x), ..., t(x)) \ dx$; φ is an arbitrary phase of the motion. From the relation (7.6), the variable t can be obtained as a function of the general coordinate x, which corresponds to the principal concept of the Rauscher method. Then solution (7.6) is substituted into Eq. (7.1); the new pseudo-autonomous dynamical system is obtained, which is solved by the above-presented approach. Thus, the iterative procedure for the NNM calculation in the nonconservative system is obtained. The details of this approach are discussed in publications [14,15,24,25].

In the case of the nonautonomous system (7.1), the determination of the steady-state resonance motions in the form of the NNMs must be completed by the following periodicity condition:

$$T + \varphi = \frac{1}{\sqrt{2}} \oint \frac{d\xi}{\sqrt{h(\varepsilon) - \Pi(\xi, x_2(\xi), ..., x_n(\xi)) - \varepsilon F(\xi, \varepsilon)}} \tag{7.7}$$

where T is a period of the external excitation; the integration is carried out during the period of the steady-state motions.

The terms $f_i(x_1, \dot{x}_1, x_2, \dot{x}_2, ..., x_n, \dot{x}_n, t)$ of the system (7.1) can lead to the self-excited vibrations. Then the above-considered approach is supplemented by the equation [25]

$$\oint f_1(\xi, \dot{\xi}(\xi), x_2(\xi), \dot{x}_2(\xi), ..., x_n(\xi), \dot{x}_n(\xi)) d\xi = 0 \tag{7.8}$$

The condition (7.8) means that a loss of energy in average over the period of the motions solutions is absent. This condition can be named the "potentiality condition."

7.2 Forced Vibrations of Two-DOF System.
As an example, the suggested methodology of forced vibrations analysis is applied to the following two-DOF system [15]:

$$\ddot{x}_1 + x_1 + x_1^3 + K_1(x_1 - x_2) + K_3(x_1 - x_2)^3 = \varepsilon \ p(t)$$
$$\ddot{x}_2 + x_2 + x_2^3 + K_1(x_2 - x_1) + K_3(x_2 - x_1)^3 = 0 \tag{7.9}$$

For periodic steady-states, the following pseudo-autonomous system is satisfied:

$$\ddot{x}_1 + x_1 + x_1^3 + K_1(x_1 - x_2) + K_3(x_1 - x_2)^3 = \varepsilon \ \hat{p}(x_1);$$
$$\ddot{x}_2 + x_2 + x_2^3 + K_1(x_2 - x_1) + K_3(x_2 - x_1)^3 = 0 \tag{7.10}$$

So, in correspondence with the principal idea by the Rauscher method, the function $p(t)$ must be changed by the function $\hat{p}(x_1)$. The NNMs of the obtained autonomous system (7.10) are represented by the modal line $x_2 = \hat{x}_2(x_1)$, governed by the following equation:

$$-2\hat{x}_2''\{0.5(x_1^2 - X_1^2)(1 + K_1) + 0.25(x_1^4 - X_1^4)$$

$$+ \int_{X_1}^{x_1}\{K_3[\xi - \hat{x}_2(\xi)]^3 - K_1\hat{x}_2(\xi) - \varepsilon\,\hat{p}(\xi)\}\,d\xi\}$$

$$- \hat{x}_2'\{x_1 + x_1^3 + K_1x_1 - \hat{x}_2K_1 + K_3(x_1 - \hat{x}_2)^3 - \varepsilon\,\hat{p}(x_1)\}$$

$$+ \hat{x}_2 + \hat{x}_2^3 + K_1\hat{x}_2 - K_1x_1 + K_3[\hat{x}_2 - x_1]^3 = 0 \qquad (7.11)$$

where X_1 is an amplitude of the vibrations. This equation must be supplemented by the boundary condition (7.4), which can be presented as

$$\hat{x}_2'(X_1)\{X_1 + X_1^3 + K_1X_1 - K_1\hat{x}_2(X_1) + K_3[X_1 - \hat{x}_2(X_1)]^3 - \varepsilon\hat{p}(X_1)\}$$

$$+ \hat{x}_2(X_1) + \hat{x}_2^3(X_1) + K_1\hat{x}_2(X_1) - K_1X_1 + K_3[\hat{x}_2(X_1) - X_1]^3 = 0$$
$$(7.12)$$

Solution of the Eq. (7.11) has the following form: $\hat{x}_2 = \hat{x}_2^{(0)} + \varepsilon\hat{x}_2^{(1)} + O(\varepsilon^2)$. A zero order approximation $\hat{x}_2^{(0)}(x_1)$ corresponds to the similar NNMs of the unforced system ($\varepsilon = 0$): $\hat{x}_2^{(0)}(x_1) = cx_1$; $c = \pm 1$. In order to determine $x_1(t)$ the single-DOF nonlinear system is analyzed. The zero order approximation of the time response is the following:

$$x_1(t) = X_{10}cn(qt, k) = X_{10}\cos\phi \qquad (7.13)$$

where cn is the elliptic cosine; X_{10} is a zero approximation of the amplitude X_1; $\phi = am(qt, k)$. The parameters q and k are presented in the Ref. [15]. The inversion of the zero order solution (7.13) is the following: $t = t(x_1) = F\{\sin^{-1}[1 - (x_1/X_{10})^2]^{1/2}, k\}/q$. So, $\hat{p}_0(x_1) = p[t(x_1)] = p[F\{\sin^{-1}[1 - (x_1/X_{10})^2]^{1/2}, k\}/q]$.

The solution of the next approximation by $\varepsilon\,\hat{x}_2^{(1)}(x_1)$ can be presented as

$$\hat{x}_2^{(1)} = a_{21}^{(1)}x_1 + a_{23}^{(1)}x_1^3 + a_{25}^{(1)}x_1^5 + \dots$$
$$= a_{21}^{(1)}X_{10}\cos\phi + a_{23}^{(1)}X_{10}^3\cos^3\phi + a_{25}^{(1)}X_{01}^5\cos^5\phi + \dots$$

The excitation is a periodic function of ϕ, which is presented as a Fourier series

$$\hat{p}(\phi_1) = \sum_{n=0}A_n\cos n\phi + \sum_{m=1}B_m\sin m\phi \qquad (7.14)$$

The expressions (7.14), (7.13), and (7.12) are substituted into (7.10) and (7.11) and the coefficients of the respective powers of $\cos\phi$ and $\sin\phi$ are matched. So, the system of linear algebraic equations with respect to $a_{21}^{(1)}, a_{23}^{(1)}, a_{25}^{(1)}, \dots$ can be derived. The solution of this system is presented in Ref. [15]. Thus, the first approximation of the NNM is obtained. It is possible to construct the highest approximations too. Details of this procedure are presented in Ref. [15].

7.3 Iterative Rauscher Method for Forced Vibrations Analysis. In the Secs. 7.1 and 7.2, the combination of the Kauderer–Rosenberg NNMs and the Rauscher method are used to analyze the forced vibrations. Now the combination of the Shaw–Pierre NNMs and the Rauscher method are treated. This approach is developed in Refs. [26, 137]. The following dynamical system with respect to the modal coordinates is considered:

$$\ddot{x}_i + \omega_i^2 x_i + R_i(x_1, \dots, x_n) = h_i\cos(\Omega t); \ i = 1, \dots, n \qquad (7.15)$$

The functions R_i describes the nonlinear terms, which contains the second and highest degrees of the general coordinates. It is

assumed that the eigenfrequencies ω_i do not satisfy conditions of the internal resonances and Ω are close to eigenfrequency ω_l. In the region of the main resonance, vibrations amplitudes of the general coordinate q_l are higher than the amplitudes of the rest general coordinates. The iterative procedure is used to analyze the forced resonance vibrations. On the first iterations, it is assumed that $q_l \neq 0$ and $q_m = 0$ ($m \neq l$). Then the system (7.15) is transformed into the following single-DOF system:

$$\ddot{q}_l + \omega_l^2 q_l + \tilde{R}_l(q_l) = h_l\cos(\Omega t) \qquad (7.16)$$

where $\tilde{R}_l(q_l) = R_l(0, \dots, 0, q_l, 0, \dots, 0)$. Following to the principal idea of the Rauscher method, the solution of the system (7.16) is presented as

$$\cos\Omega t = r(q_l) = \alpha_0 + \alpha_1 q_l + \alpha_2 q_l^2 + \cdots \qquad (7.17)$$

The approach for determination of the series (7.17) is developed in Refs. [26, 137]. The series (7.17) are substituted into the system (7.15). As a result, the pseudo-autonomous dynamical system is obtained

$$\ddot{q}_i + \omega_i^2 q_i + R_i(q_1, \dots, q_n) = h_i(\alpha_0 + \alpha_1 q_l + \alpha_2 q_l^2 + \dots)$$
$$i = 1, \dots, n \qquad (7.18)$$

If the pseudo-autonomous dynamical system (7.18) has equilibria $q_{0,i}$, the following change of variables is applied to the system (7.18): $q_i = q_{0,i} + \eta_i(t)$; $i = 1, \dots, n$. Introducing the modal general coordinates $(\xi_1, \dots, \xi_n)$ of the linearized system, the following dynamical system is derived:

$$\ddot{\xi}_i + v_i^2\xi_i + L_i(\xi_1, \dots, \xi_n) = 0; \quad i = 1, \dots, n \qquad (7.19)$$

where L_i is the nonlinear part of the dynamical system.

The Shaw–Pierre NNMs approach is applied to the autonomous system (7.19). Then the obtained solution can be substituted into (7.15). Thus, the single-DOF dynamical system is derived; the iterative loop of the Rauscher method can be continued.

The suggested method of forced vibrations analysis can be applied to analyze parametric vibrations of essential nonlinear systems, which is treated in Ref. [26].

7.4 Pierre–Shaw Method for Forced Vibrations Analysis. Other method for forced vibrations analysis is suggested in the paper [140]. The mechanical systems accounting dissipation are considered in the following form:

$$\ddot{x}_i + \beta_i\dot{x}_i + \omega_i^2 x_i + R_i(x_1, \dots, x_n) = h_i\cos\phi_f; \ i = 1, \dots, n$$
$$\phi_f = \Omega t + \tilde{\phi}_0 \qquad (7.20)$$

In order to obtain the forced vibrations of the system (7.20), the invariant manifolds are considered. Then this system is treated in $(2N + 1)$ phase space:

$$\dot{x}_i = y_i$$
$$\dot{y}_i + \beta_i y_i + \omega_i^2 x_i + R_i(x_1, \dots, x_n) = h_i\cos\phi_f; \ i = 1, \dots, n \qquad (7.21)$$
$$\dot{\phi}_f = \Omega$$

The n pairs of variables (x_i, y_i) are divided into two separate groups: master coordinates and slave coordinates. The pair of the state variables, whose modal frequency ω_k is close to the excitation frequency Ω, is retained as a master coordinate. Thus, the primary resonance of the nonlinear system is considered. The master coordinates (x_k, y_k) are transformed into the polar ones: $(x_k, y_k) = a(\cos\phi, -\omega_k\sin\phi)$. The variables (a, ϕ) can be determined from the following system:

$$\dot{a} = -2\beta_k a \sin^2\phi - \left(h_k \cos\phi_f - R_k\right) \omega_k^{-1} \sin\phi$$

$$\dot{\phi} = \omega_k - \beta_k \sin^2\phi - a^{-1}\omega_k^{-1}\left(h_k \cos\phi_f - R_k\right)\cos\phi \qquad (7.22)$$

$$\dot{\phi}_f = \Omega$$

The slave coordinates depend on the master coordinates in the following form:

$$x_i = P_i\left(a, \phi, \phi_f\right); \quad y_i = Q_i\left(a, \phi, \phi_f\right); \quad i = 1, ..., n; \quad i \neq k \qquad (7.23)$$

The invariant manifold (7.23) satisfies the following system of PDE:

$$Q_i = \frac{\partial P_i}{\partial a}\left[-2\beta_k a \sin^2\phi - \left(h_k \cos\phi_f - R_k\right)\omega_k^{-1}\sin\phi\right] + \frac{\partial P_i}{\partial \phi}$$

$$\times \left[\omega_k - \beta_k \sin 2\phi - \left(h_k \cos\phi_f - R_k\right) a^{-1}\omega_k^{-1}\cos\phi\right] + \frac{\partial P_i}{\partial \phi_f}\Omega;$$

$$-2\beta_i Q_i - \omega_i^2 P_i - R_i + h_i \cos\phi_f = \frac{\partial Q_i}{\partial a}\left[-2\beta_k a \sin^2\phi\right.$$

$$-\left(h_k \cos\phi_f - R_k\right)\omega_k^{-1}\sin\phi\right] + \frac{\partial Q_i}{\partial \phi}[\omega_k - \beta_k \sin 2\phi$$

$$-\left(h_k \cos\phi_f - R_k\right) a^{-1}\omega_k^{-1}\cos\phi] + \frac{\partial Q_i}{\partial \phi_f}\Omega$$

This method is applied to analyze the forced vibrations of two-DOF mechanical system in the paper [140].

7.5 Principal Trajectories in Nonautonomous Systems.
Zhuravlev [141] suggested to seek normal mode solutions and defined them as "principal directions." It was assumed that the linear n-DOF forced system oscillates as a single harmonic oscillator so, that the vector of the general coordinates $x(t)$, and the force vector $p(t)$ are collinear to the constant vector q, that is $x = q\cos(\omega t); p = \mu q\cos(\omega t)$. In the case of forced vibration, the predetermined forced frequency ω should not play the role of eigenvalue. However, the coefficient of proportionality μ can be chosen instead of the eigenvalue. In the nonlinear case, the definition of Zhuravlev is not applicable so that the above term "directions" loses its global sense. The principal "directions" can be replaced by "trajectories." The related vibration and forcing are neither harmonic in time nor similar in configuration space [142]. Principal trajectories of forced vibrations of the linear and nonlinear systems are defined by Pilipchuk as motions in which the nonconservative system is equivalent to a Newtonian particle in the configuration space. A trajectory of the nonlinear nonautonomous system in which the system is equivalent to a Newtonian particle (that is, the acceleration and the external force vectors are coupled by the Newton's second law ($m\ddot{x}(t) = p(\omega t)$), is called the principal trajectory of the forced vibration. The proposed definition is applied to analyze the single-DOF Duffing oscillator, when the periodic solution up to $O(\varepsilon^2)$ is obtained easy. Moreover, this definition is used to study two DOF quasilinear system under the action of periodic excitation. The conception of the principal trajectories in the nonautonomous systems is generalized to continuous systems too. The continuous system performing forced vibrations is equivalent to a Newtonian particle in the function space of configurations, that is the acceleration and the external forcing vectors satisfy the Newton's second law $\sigma\left(\partial^2 u(t, y)/\partial t^2\right) = p(\omega t, y)$. Such system motions are called principal modes of forced vibration. As an example, the dynamics of string with the concentrated masses is considered. It is assumed that the masses are supported by nonlinear springs with cubic characteristic.

7.6 Two-Degree-of-Freedom System With Time Delay.
Gendelman [143] analyzed NNMs in two-DOF dynamical system with time delay. The following essentially nonlinear system is treated:

$$\ddot{y}_1 + y_1^m + k(y_1 - y_2(t-1))^m = 0$$
$$\ddot{y}_2 + y_2^m + k(y_2 - y_1(t-1))^m = 0 \qquad (7.24)$$

where m is an odd positive number; time delay is equal to 1. NNMs of the system (7.24) are presented as

$$y_1 = cy_2(t - T) \qquad (7.25)$$

where c is a ratio of amplitudes; T is the constant phase shift. Solutions with the period Δ are considered. Due to symmetry of the equations of motions, the following conditions take place:

$$y_2 = y_2(t + \Delta); \quad y_2 = -y_2\left(t + \frac{\Delta}{2}\right)$$

The motions (7.25) take place if the next equalities are satisfied

$$T - 1 = 0.5\,n\,\Delta; \quad T + 1 = 0.5\,l\,\Delta; \quad n, l \in Z$$

The following cases are considered:

(a) Both l and n are even, and the cases $\Delta = 2/q$; $q = 1, 2, ...$ and $T = 0$ or $T = 0.5\,\Delta$ are considered. Then the solutions (7.25) take place in the system (7.24) if the following equation is satisfied: $1 + k\,(1 - c)^m = c^{m-1} + kc^{-1}(c - 1)^m$. This equation has solutions $c = \pm 1$, corresponding to the symmetric and antisymmetric NNMs. At a certain critical value of k, the solutions (7.25) bifurcate giving rise to localized modes.

(b) Both l and n are odd. This case can be reduced to the previous one.

(c) l is even and n is odd. In this case, it is taken $\Delta = 4/(2q + 1)$; $q = 0, 1, 2, ...$ and $T = 0.25\,\Delta$, or $T = 0.75\,\Delta$. The motions correspond to ovals in configuration space.

(d) l is odd and n is even. It is easy to show that a sign inversion of c bring this case to (c).

So, one can conclude that the system (7.24) possesses two families of synchronous periodic solutions. The first family corresponds to the cases (a) and (b); it can be presented by straight modal lines. The second one corresponds to the cases (c) and (d); it is described by ovals in the configuration plane.

8 Numerical Method for NNMs Calculations

The shooting method can be used to analyze periodic motions of the dynamical system with arbitrary degrees-of-freedom [144]. In particular, two-dimensional shooting diagrams are presented for visualization of manifolds of periodic solutions and their bifurcations in Ref. [59]. Here, the shooting method is presented to calculate NNM in the configuration space. This method is suggested in the paper [84]. In order to explain this approach the two degrees-of-freedom autonomous system of the form (2.1) with general coordinates x_1, x_2 is considered. The equation of the modal lines $x_2(x_1)$ is written as the first order ordinary differential equations

$$z_1' = f_1(z_1, z_2, x_1) = z_2$$

$$z_2' = f_2(z_1, z_2, x_1) = \frac{\left(\Pi_{X_1} z_2 - \Pi_{Z_1}\right)\left(1 + z_2^2\right)}{2(h - \Pi)} \qquad (8.1)$$

where $z_1 = x_2$; $z_2 = x'_2$. Two functions are introduced

$$
\begin{aligned}
G_1(z_1, x_1) &= h - \Pi(x_1, z_1(x_1)) \\
G_2(z_1, z_2, x_1) &= \Pi_{Z_1}(x_1; z_1(x_1)) - \Pi_{X_1}(x_1, z_1(x_1))z_2(x_1)
\end{aligned}
\tag{8.2}
$$

Two boundary conditions (2.8) describing the intersections of modal lines with the maximal equipotential surface in the points $x_1 = X_1$ and $x_1 = X_2$ can be presented as

$$
\begin{aligned}
G_1(z_1(X_i), X_i) &= 0 \\
G_2(z_1(X_i), z_2(X_i), X_i) &= 0 \; ; \; i = 1, 2
\end{aligned}
\tag{8.3}
$$

Thus, a determination of NNMs in the configuration space is reduced to the solution of the system of differential equation (8.1) with boundary conditions (8.3). The points $x_1 = X_1$ and $x_1 = X_2$ can not be taken as initial conditions for the numerical solution because they are singular points of the Eq. (8.1). Numerical solution of the system (8.1) starts from $x_1 = 0$. In general case modal lines do not pass through origin of coordinates. So, in this point initial conditions for z_{10} and z_{20} are not known. It is assumed that at $x_1 = 0$ the parameters z_{10}, z_{20} are set arbitrarily. Then as a result of the numerical integration the values $z_1(X_i)$; $z_2(X_i)$; $i = 1, 2$ are obtained. In general, these values do not satisfy the boundary conditions (8.3). Therefore, the parameters z_{10}, z_{20} are adjusted as the values $z_1(X_i)$; $z_2(X_i)$; $i = 1, 2$ satisfy the boundary conditions (8.3). The deep treatment of this approach can be found in the paper [84].

The numerical approach for determination of the Shaw–Pierre NNMs is suggested in Refs. [145,146]. The continuation procedure, named by the asymptotic numerical method, with the harmonic balance method is used to construct of the Shaw–Pierre NNMs in Ref. [147]. Three numerical approaches to determine the NNMs in two DOF system with nonlinear spring are compared in Ref. [148]. The Shaw–Pierre approach, the Belizzi and Bouc formulation, and the Lewandowski method [149] are considered. It is stressed that the last method is more effective for large vibration amplitudes and bifurcations analysis.

Semi-analytical approach of the NNMs stability analysis is proposed in Ref. [150]. This approach is based on the definition of Lyapunov stability. The forced vibration modes in systems with several equilibrium positions are considered. The stability of the NNMs with both regular and chaotic behavior in time is analyzed.

9   Dynamics of Piecewise Linear and Impact Systems

Spatial symmetries of motions, their stability, and localization of two- and three-DOF impact systems with bilateral perfectly rigid barriers were considered in Ref. [30]. NNMs in two-DOF impact system with sufficiently small elastic coupling are considered by Mikhlin et al. [151]. Trajectories of the vibro-impact modes are obtained in power series by the small coupling parameter and by the one general coordinate. It is shown that double-sided vibro-impact localized motions are orbitally stable for high energy of the system. When the energy is decreased, the double-sided impact motions transform to the single-sided ones, and the additional boundary condition (2.8) must be used. The global vibro-impact response of the system is obtained by numerical computations of Poincare maps. Besides, the inverse problem for the vibro-impact modes is considered. Namely, the trajectories of the modes are given as polynomials, and a potential of the vibro-impact systems possessing these modes is determined.

The nonsmooth time transformation by Pilipchuk is used to investigate the NNMs in two-DOF impact systems in Ref. [152]. This transformation creates singularities points, which can be eliminated by additional boundary conditions. Three types of periodic solutions are treated and three types of the corresponding boundary conditions are formulated. Periodic motions with one- and two-sided impacts are analyzed numerically. Typical trajecto-

ries of such solutions in the configuration space are shown. A class of strongly nonlinear many DOF vibrating systems including bilateral rigid barriers is considered in Ref. [32]. Nonlinear impact motions are obtained using new saw-tooth time transformation. As a result, the vibro-impact system is transformed to the nonautonomous one and the periodic solution in a form of impact mode can be obtained analytically. Based on this special representation, sufficient conditions of existence of impact motions are considered by using the spectral axes. It is shown that for high frequency vibrations the impact modes become localized spatially and always exist. Systems with multiple impacting particles are also considered. This approach is used for finite-DOF system with synchronous impacts on stiff constraints in Ref. [153]. It is shown that a specific combination of two impact motions results in another impact mode. Perfectly localized impact motions are described by the high-energy asymptotics. Results on the impact motions are summarized in the book by Pilipchuk [59]. It is also shown that a specific combination of two impact motions gives another impact mode. The corresponding manipulations with impact modes become possible due to the availability of closed form exact solutions obtained by means of the triangular sine temporal transformation for impulsively loaded and vibro-impact systems. In particular, the idea by Van-der-Pol and averaging tool are adapted for the case of impact oscillator. Nonsmooth spatial unfolding transformation is used jointly the Van-der-Pol method to analyze single-DOF impact system in Ref. [154]. The Lie series is used to construct periodic solutions (in particular, NNMs) in the vibro-impact systems [155].

The autonomous piecewise linear multi-DOF gyroscopic and nongyroscopic systems are studied in Ref. [31]. The NNMs are presented as Fourier series. It is shown that the gyroscopic effects lead to appearance of the vibration modes with closed trajectories in the system configuration space. The bifurcated modes with curvilinear trajectories are obtained and they are calculated numerically in points of the NNMs bifurcations. These points are obtained by calculation of characteristic multipliers. The NNMs in piecewise linear two-DOF systems are studied by asymptotical and numerical method in Ref. [156]. Chen and Shaw [157] suggested the asymptotic method to construct Poincare' sections of piecewise linear n-DOF systems close to vibro-impact. Jiang et al. [158] developed NNMs, which are considered in Sec. 6, to analyze piecewise linear systems.

10   NNMs for Continuous Systems

10.1   Shaw–Pierre Approach. The generalization of the central manifold theory for ordinary differential equations on distributed systems is treated in the book [37]. Determination of both the Shaw–Pierre NNMs and the central manifolds is very close. Therefore, a generalization of the NNMs to continuous mechanical systems is possible. In future consideration, we follow the paper [18].

Continuous mechanical systems, which occupies one-dimensional region Ω, are considered. Displacements and velocities of the system arbitrary point are denoted by $u(x, t)$ and $v(x, t) = \partial u/\partial t$, respectively, where x is the system position coordinate. The partial differential equations of the system motion can be presented as

$$
\begin{aligned}
\frac{\partial u(x, t)}{\partial t} &= v(x, t) \\
\frac{\partial v(x, t)}{\partial t} &= F[w(x, t); v(x, t)] \; ; \; x \in \Omega
\end{aligned}
\tag{10.1}
$$

where F is a nonlinear operator. In general, this operator is integrodifferential. The boundary conditions of the considered mechanical system are presented in general form as

$$
B(u(x, t), v(x, t)) = 0 \; ; \; x \in \partial\Omega
\tag{10.2}
$$

where B is an operator of the boundary conditions; $\partial\Omega$ is a boundary of region Ω.

Following the paper [18], the Shaw–Pierre continuous nonlinear modes of autonomous systems (10.1) are determined as motions, which belong to two-dimensional invariant manifold, which has the following properties. They pass through the stable equilibrium position $(u,v) = (0,0)$; they are tangent to the subspace of the eigenmodes of linearized system. Key idea of the continuous nonlinear modes is expressed as the following. If displacements and velocities of some point $x \equiv x_0$ are known $u_0(t) = u(x_0, t)$; $v_0(t) = v(x_0, t)$, then the displacements and velocities of the rest points of continuous system can be determined through $u_0(t)$; $v_0(t)$. This is expressed by the next relations

$$u(x,t) = U(u_0(t),\ v_0(t), x)$$
$$v(x,t) = V(u_0(t),\ v_0(t), x) \tag{10.3}$$

Equations (10.3) are described a manifold in infinite-dimensional phase space. Due to existence of such manifold, continuous mechanical system is reduced to one-DOF oscillator. The next relations are true

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial u_0}\dot{u}_0 + \frac{\partial U}{\partial v_0}\dot{v}_0; \qquad \frac{\partial V}{\partial t} = \frac{\partial V}{\partial u_0}\dot{u}_0 + \frac{\partial V}{\partial v_0}\dot{v}_0 \tag{10.4}$$

The system of partial differential equations with respect to continuous nonlinear mode is the following:

$$V = \frac{\partial U}{\partial u_0}v_0 + \frac{\partial U}{\partial v_0}F[U,V]|_{X=X_0}$$
$$F[U,V] = \frac{\partial V}{\partial u_0}v_0 + \frac{\partial V}{\partial v_0}F[U,V]|_{X=X_0};\ x \in x_0 \tag{10.5}$$

The boundary conditions are added to the system (10.5)

$$B(U,V) = 0;\ x \in \partial\Omega \tag{10.6}$$

It is possible to obtain the continual nonlinear mode of the system (10.5) in power series by the variables u_0, v_0

$$\begin{aligned}
U(u_0(t),\ v_0(t), x_0) &= a_1(x, x_0)u_0(t) + a_2(x, x_0)v_0(t) \\
&\quad + a_3(x, x_0)u_0^2(t) \\
&\quad + a_4(x, x_0)u_0(t)v_0(t) + ... \\
V(u_0(t),\ v_0(t), x_0) &= b_1(x, x_0)u_0(t) + b_2(x, x_0)v_0(t) \\
&\quad + b_3(x, x_0)u_0^2(t) \\
&\quad + b_4(x, x_0)u_0(t)v_0(t) + ...
\end{aligned} \tag{10.7}$$

The expansions (10.7) are substituted into the Eq. (10.5) and the boundary conditions (10.6). Expansions in terms of u_0 and v_0 are performed and the coefficients at $u_0^m v_0^n$; $(m, n) = 0, 1, 2, ...; m+n \geq 1$ are matched. The detailed treatment of this procedure is presented in Ref. [18].

If the manifold is obtained, the functions (10.7) are substituted into the equations of the continuous system (10.5) to determine the system dynamics on manifold. As a result, the one-DOF autonomous dynamical system with respect to the variables (u_0, v_0) is derived

$$\frac{du_0}{dt} = v_0;\qquad \frac{dv_0}{dt} = F[u(x,t),\ v(x,t)]|_{X=X_0}$$

This equation describes the dynamics of the continuous system in the point $x = x_0$.

10.2 King–Vakakis Approach. In order to describe the King–Vakakis approach [35], bending vibrations of a beam on nonlinear elastic foundation are considered. The equation of the

system vibrations with respect to the dimensionless parameters has the following form:

$$w_{tt} + w_{xxxx} + kw + \varepsilon\gamma w^3 = 0 \tag{10.8}$$

where $w(x, t)$ is the beam dynamic deflection. The main limitation of this method is a separation of variables in unperturbed system ($\varepsilon = 0$). The potential and kinetic energies per unit of length are the following:

$$2\Pi = w_{xx}^2 + kw^2 + \frac{\varepsilon\gamma}{2}w^4$$
$$2T = w_t^2 \tag{10.9}$$

The Lagrangian per unit of length is the following: $2L = w_t^2 - w_{xx}^2 - kw^2 - 0.5\,\varepsilon\gamma w^4$. The considered system is conservative with the following energy integral:

$$E_* = 0.5 \int_0^1 \left(w_t^2 + w_{xx}^2 + kw^2 + 0.5\,\varepsilon\gamma\,w^4\right)dx \tag{10.10}$$

where the integral is taken over all length of the beam. Displacements of an arbitrary beam point $x = x_0$ are chosen as a general coordinate:

$$u(t) = w(x_0, t)$$

Following the definition of Shaw and Pierre [18,159], if displacements of all system points vanish and reach extreme values simultaneously, then these motions are called continuous NNMs. The displacements of all system points in NNMs are a function of one point motions of the continuous system $u(t)$

$$w(x, t) = W(x, u(t)) \tag{10.11}$$

where $W(x, u)$ is called the modal function. Continuous NNMs are a generalization of the Kauderer–Rosenberg NNMs in finite-DOF systems, which are considered above. The beam deflection of a continuous nonlinear mode can be described by the function $W(x, u)$, which has not explicit dependence on t. Note, that the function $W(x, u)$ exists, if the point $x = x_0$ is not a nodal point. The following relations follow from (10.11):

$$\frac{\partial w}{\partial t} = \frac{\partial W}{\partial u}\dot{u};\qquad \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 W}{\partial u^2}\dot{u}^2 + \frac{\partial W}{\partial u}\ddot{u} \tag{10.12}$$

Substituting (10.12) into (10.10), it is derived:

$$\dot{u}^2 = \frac{2E_* - \int_0^1\left(W_{xx}^2 + kW^2 + 0.5\varepsilon\,\gamma W^4\right)dx}{\int_0^1 W_u^2 dx} \tag{10.13}$$

As a result, the beam continuous NNMs can be determined by the following equation:

$$\frac{\partial^2 W}{\partial u^2}\frac{2E_* - \int_0^1\left(W_{xx}^2 + k\,W^2 + 0.5\varepsilon\,\gamma\,W^4\right)dx}{\int_0^1 W_u^2 dx}$$
$$- \frac{\partial W}{\partial u}\left(W_{xxxx} + k\,W + \varepsilon\,\gamma\,W^3\right)_{x=x_0} = -W_{xxxx} - k\,W - \varepsilon\,\gamma\,W^3 \tag{10.14}$$

The maximal equipotential surface, where $\dot{u} = 0$, is obtained from the following relation: $\int_0^1\left(W_{xx}^2 + kW^2 + 0.5\varepsilon\gamma W^4\right)dx = 2E_*$. The

respective values of the amplitudes are $u = u_*$. As the conservative system is considered, one of the values u_*, E_* can be taken arbitrary. Evidently, the points of maximal equipotential surface are singular points of the Eq. (10.14). By analogy with the theory of the NNMs of finite-DOF systems, these singularities are removed, and the analytical continuation of the solution to maximal equipotential surface is possible, if the following boundary condition is satisfied:

$$\left[-\frac{\partial W}{\partial u} \left(W_{xxxx} + kW + \varepsilon\gamma W^3 \right)_{x=x_0} + W_{xxxx} + kW + \varepsilon\gamma W^3 \right]_{u=u_*} = 0$$
(10.15)

The modal function is presented in the following form:

$$W(x, u(t)) = W_0 + \varepsilon W_1 = \psi_0(x)u(t) + \varepsilon \sum_{m=1,3,\dots} \psi_m(x)u^m(t)$$
(10.16)

The detailed treatment of this construction can be found in the paper [35].

10.3 Other Approaches. The generalized Ritz method and the NNMs conception are used to study forced vibrations of nonlinear elastic systems with nonlinear boundary conditions in Ref. [160]. Localization of NNMs in continuous system is investigated in the paper [161]. The definition of strong and weak localization of continuous system is suggested.

Authors of the paper [36] reformulated the invariant manifold approach in a complex framework and proposed a computationally efficient extension of this methodology. This approach can be extended for computation of higher-dimensional invariant manifolds for NNMs of the systems with internal resonance [162,163]. Coupled and uncoupled resonant NNMs have been studied for two-to-one, three-to-one and one-to-one internal resonances by Lacarbonara et al. [131,164]. The multiple-scale method has been applied to the Galerkin-reduced model or directly to the equations of motion and boundary conditions of a general class of one-dimensional continuous systems with weak quadratic and cubic nonlinearities. Existence, stability and orthogonality of these modes are investigated. This approach is used to study vibration modes of quasilinear systems with dual internal resonances in Ref. [115].

Andrianov [165] connected the NNMs of continuous systems with so-called intermediate asymptotics of Barenblatt and Zel'dovich [166]. The similar idea is used by Bolotin [167]. The basic idea of this approach is a construction of certain particular self-similar solutions of a nonlinear problem, which can be considered as asymptotics of a wide class of other solutions. A realization of this approach for nonlinear dynamics of elastic systems (beams, plates, shallow shells) is presented by Andrianov and his co-authors [168–172]. The paper [172] is a survey of publications on the NNMs and similar solutions in nonlinear continuous elastic systems.

11 Localized NNMs

One of the most interesting features of NNMs is that they may induce nonlinear mode localization in dynamical systems, i.e., a subset of NNMs may be spatially localized to subcomponents of dynamical systems. It is known a phenomenon of existing of localized modes in linear systems. In this case, there are only small number excited substructures; the remaining structures are motionless. The majority of studies devoted to localization deal with linear vibrations. Pierre and Dowell [173] used a perturbation methodology to study the mode localization in a system containing weakly coupled, weakly mistuned, and linearized pendulums. Review on localization in linear systems is presented in the paper [174]. Note, that the most studies in this subject devoted to repetitive cyclic structures consisting of nearly identical interconnected elements. Analysis have shown that the localization in linear periodic systems takes place when

(1) the coupling between structures is sufficiently weak
(2) perfect periodicity is perturbed by small structural mistuning

Nonlinear mode localization has been analyzed by using the concept of NNMs. The results of Vakakis and his co-authors [175,176] show that the localized NNM exists for weakly coupled, nonlinear cyclic structures consisting of finite-DOF oscillators. The free oscillations of **n**-DOF nonlinear systems with cyclic symmetry and weak coupling between substructures are examined in Ref. [175]. It is shown that the NNM localization occurs in the perfect symmetric, weakly coupled structure. Investigations of Vakakis [177] and King and Vakakis [178] show an existence of the nonlinear mode localization of flexible structures in the absence of structural disorders. In Ref. [161], the authors considered a possibility of the localization in coupled beams on the nonlinear foundations, by using the concept of the NNMs for continuous systems and the energetic estimations for the NNMs.

The dynamics of weakly coupled, nonlinear cyclic assemblies are investigated in the presence of weakly structural mistuning in the paper [179]. Weak structural mistuning of the order $O(\varepsilon)$ is introduced in the masses, coupling and ground stiffness of each oscillator. The multiple-scale method is used to study nonlinear dynamics of such system. The primary focus is to quantify the combined influence of structural irregularity and nonlinearity on the underlying modal structure of the cyclic system. The spatial distribution of vibration energy is found to be highly dependent on the amount of structural irregularity presented in the system. The unforced cyclic assembly exhibits a combination of localized and nonlocalized NNMs, exceeding the number of modes predicted by linear theory.

Monocoupled periodic systems with infinite degrees-of-freedom and with one or two nonlinear subsystems are considered in Ref. [180]. The effect of the nonlinear disorders on localized motions is examined by using the Lindstedt–Poincare method. The key to solve the problem lies in solving the set of nonlinear algebraic equations with infinite number of variables, which is obtained from the system of modulation equations. The closed-form solutions of nonlinear algebraic equations describing the localized modes are found by means of U-transformation technique. Depending on the value of vibrations amplitudes, different number of NNMs is observed in these systems. The following classes of NNMs are obtained depending on the vibrations amplitudes: one antisymmetric stable localized mode; three localized modes in which two modes are nonsymmetrical and stable while the other one is antisymmetric and unstable; four localized modes in which two nonsymmetric unstable and one symmetric stable mode.

Two-degree-coupling spatially periodic system with infinite number of subsystems and nonlinear disorder is considered in the paper [181]. The present work is based on the assumption that the nonlinear disorder and the coupling are weak and the system energy is large enough to induce the localized mode. The effect of the nonlinear disorder and the coupling stiffness on the localized modes is examined. The U-transformation to the equations of motions is used; then Lindstedt–Poincare method is applied to the resulted system. It was made the conclusion about an existence of the following steady states: strongly localized state; one symmetric localized mode; three localized mode in which two modes are nonsymmetric and stable, while the other one is symmetric and unstable.

Forced localization in the system consisting of an infinite number of coupled nonlinear oscillators is examined in the paper [182]. This system is presented as

$$\ddot{v}_n + (1 + 4\varepsilon b)v_n - \varepsilon b\left(2v_n - v_{n-1} - v_{n+1}\right) + \varepsilon\mu v_n^3 = \varepsilon(-1)^n f_n(t)$$
(10.1)

where $n = 0; \pm 1; \pm 2; ....$ A "continuous approximation" is used to reduce the infinite set of ordinary differential equations to the following nonlinear partial differential equation:

$$\frac{\partial^2 v}{\partial t^2} + \varepsilon\, b\, h^2 \frac{\partial^2 v}{\partial x^2} + (1 + 4\,\varepsilon\, b)v + \varepsilon\, \mu\, v^3 = \varepsilon(-1)^n f(x,t)$$

where h is the distance between adjacent particles. Solutions of this partial differential equation are presented as $v(x,t) = \sum_{p=1,3,5,...} A_p(x)\cos p\omega t$. The function $A_1(x)$ satisfies the following ordinary differential equation:

$$A_1'(x) - \frac{(\omega^2 - 1)}{\varepsilon\, \lambda} A_1(x) + \frac{3\mu}{4\, b\, h^2} A_1^3(x) = \frac{P(x)}{b\, h^2} , \qquad (10.2)$$

where $\lambda = bh^2/(1 + 4\varepsilon\, b)$. Thus, it is found that the amplitude modulation $A_1(x)$ satisfies a forced Duffing equation with negative stiffness. Analysis of the solution $A_1(x)$ localization is made.

The approach based on the complex representation of the dynamic equations is suggested to obtain NNMs by Manevich [86]. Two degree-of-freedom system is analyzed; localized NNM, in-phase and out-of-phase NNMs are treated.

The mode localization phenomenon is considered in a mechanical systems consisting of a simply supported beam and transverse nonlinear springs with hardening characteristics [58,59]. Based on the specific coordinate transformations and the idea of averaging, explicit equations describing the energy exchange between the local modes and the corresponding localization conditions are obtained. It was shown that when the energy is slowly pumped into the system then, at some point, the energy equipartition around the system suddenly breaks and one of the local modes becomes the dominant energy receiver.

12 The Qualitative Theory of NNMs

This review presents a constructive theory of the NNMs in nonlinear systems. But there is a significant number of publications on the qualitative NNMs theory devoted to existence of NNMs. Local motions near the origin are considered. Following Lyapunov [1], the *n*-DOF system with Hamiltonian without internal resonances has exactly *n* one-parameter families of NNMs close to stable equilibrium. These results are generalized for twice differentiated near origin Hamiltonian by Weinstein [183]. The Lyapunov's results are generalized for systems with internal resonances in [183,184]. It was shown that such systems have at least *n* one-parameter families of periodic solutions close to stable equilibrium.

Seifert [3] proved an existence at least one periodic solution of the conservative system, which trajectory twice intersects the maximal equipotential surface during a period. Later such solutions were named BB-solutions ("boundary–boundary"). Corresponding solutions passing through the origin are named BOB-solutions ("boundary–origin–boundary"). The Seifert's theorem is generalized to the systems having complex form of the kinetic energy in Ref. [185]. Theorems on existence not less than *n* BOB-solutions in the *n*-DOF system are proven for systems with even potential [186,187]. These results are obtained in supposition that the region of motion in phase space is compact, convex, and it is contained between two spheres.

An existence of the Kauderer–Rosenberg NNMs is treated by Cook and Struble [188]. The system consisting of two coupled masses with anchor nonlinear springs is examined in [189]. Under the assumption that the unperturbed system possesses the NNM having a linear modal line, a theorem guarantees that under small perturbations the NNM is a generator of a continuous family of
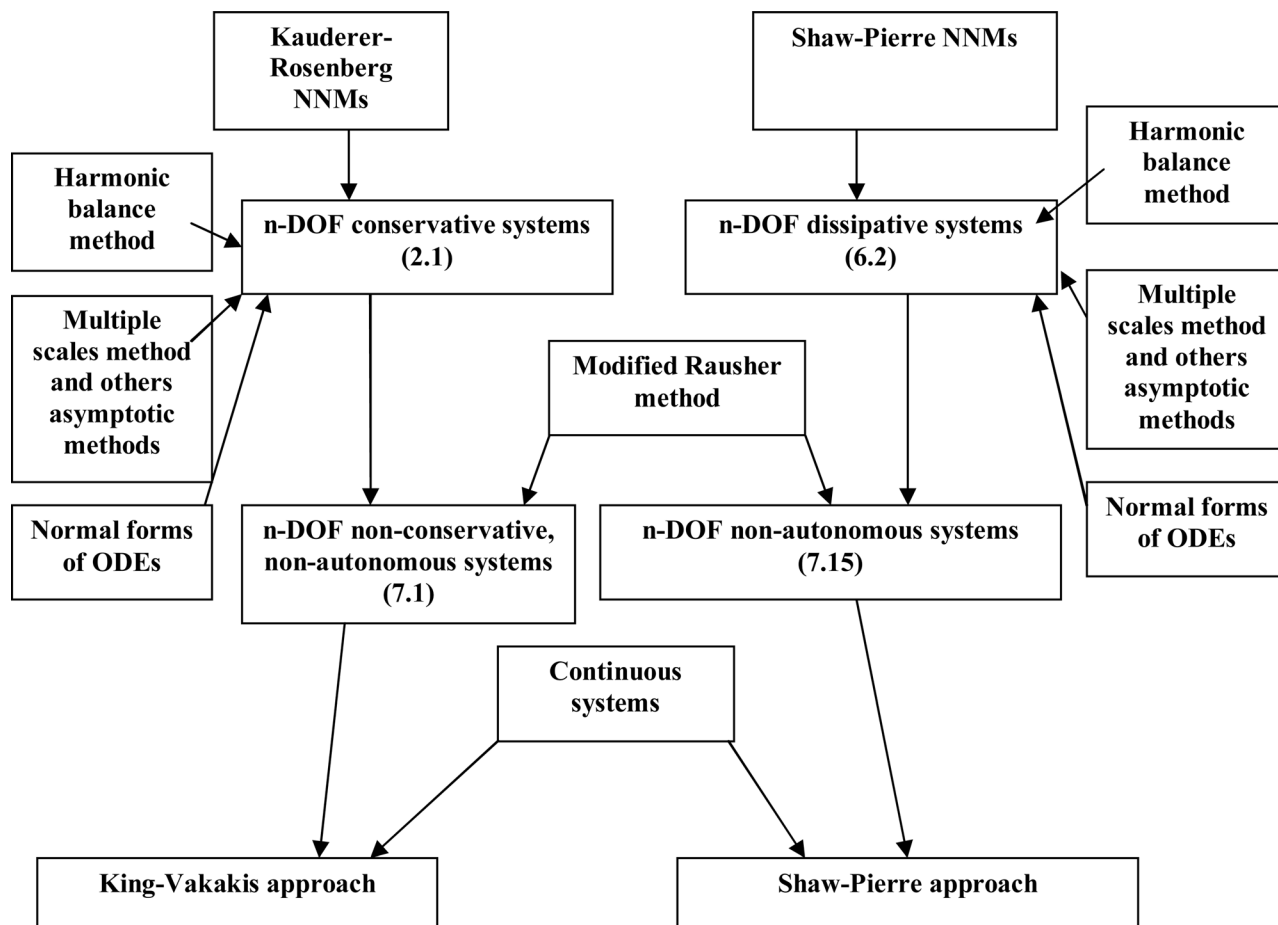

Fig. 7 Diagram of the NNM

periodic motions of the perturbed system. It is proven in Ref. [190] that the periodic solutions that pass through the origin exist in a general class of finite-DOF systems with potential energy, which is symmetric with respect to the origin of the configuration space. This result is proven by using a theory of geodesic curves in Riemann spaces. Van Groesen [65] proved the existence of at least n NNMs in n-DOF systems with homogeneous potential energy; the bifurcations of the NNMs for increased energy of vibrations are studied. Moreover, the author investigated extremal properties of the BOB-solutions. It is shown that there is periodic BOB-solution that supplies a minimum of the averaged kinetic energy.

Nonlocal criteria of existence of BOB- and BB-solutions, which is generalized the results of Lyapunov, are obtained by Zevin [15,191,192]. On the basis of the Krasnoselsky method, which is developed for the qualitative theory of differential equations, it is proven theorems, which generalize Lyapunov's results and guarantee a continuation of periodical solutions to large values of energy.

13 Conclusions

Different approaches to analyze the NNMs in conservative systems and their generalization on self-sustained vibrations, forced and parametric oscillations, are treated in this review. Abilities of these approaches application for different types of vibrations are shown in the diagram (Fig. 7). The Kauderer–Rosenberg NNMs are used, as a rule, to study free nonlinear vibrations, which are described by the system (2.1); the NNMs by Shaw–Pierre are used to investigate free nonlinear vibrations of n-DOF dissipative quasilinear mechanical system of the form (6.2). A combination of the Rausher method and the Kauderer–Rosenberg NNMs can be applied to analyze nonautonomous system of the form (7.1). Nonautonomous quasilinear n-DOF systems (7.15) can be successfully analyzed by combination of the Rausher method and the Shaw–Pierre NNMs. Both approaches, i.e., the Kauderer–Rosenberg NNMs and the Shaw–Pierre NNMs, can be used to study dynamics of nonlinear continuous systems.

It can conclude that NNMs are typical periodic regimes in different classes of nonlinear systems, which permit to determine important characteristics of these systems. Besides, the NNMs can be used as some basic regimes that allow obtain more complicated regimes of nonlinear systems.

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