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Nonlinear normal vibration modes in the dynamics of nonlinear elastic systems

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Abstract. Nonlinear normal modes (NNMs) are a generalization of the linear normal vibrations. By the Kauderer-Rosenberg concept in the regime of the NNM all position coordinates are single-values functions of some selected position coordinate. By the Shaw-Pierre concept, the NNM is such a regime when all generalized coordinates and velocities are univalent functions of a couple of dominant (active) phase variables. The NNMs approach is used in some applied problems. In particular, the Kauderer-Rosenberg NNMs are analyzed in the dynamics of some pendulum systems. The NNMs of forced vibrations are investigated in a rotor system with an isotropic-elastic shaft. A combination of the Shaw-Pierre NNMs and the Rauscher method is used to construct the forced NNMs and the frequency responses in the rotor dynamics.

1. Introduction

Kauderer [1] was a forerunner in developing quantitative methods to analyze the NNMs in two-DOF conservative systems. Rosenberg [2] defined NNMs as synchronous periodic motions during which all coordinates of the system vibrate equiperiodically, reaching their maximum and minimum values at the same instant of time. He selected a few classes of essentially nonlinear systems allowing NNMs with rectilinear trajectories (modal lines) in a configuration space. The first formulation of the NNMs can be named **the Kauderer-Rosenberg approach**. In general, the NNM modal lines are curvilinear. The power series method to construct these trajectories is proposed in [3,4]. Shaw and Pierre [5,6] proposed the other formulation of NNMs for a general class of nonlinear discrete conservative or non-conservative systems. This analysis is based on the computation of invariant manifolds on which the NNM oscillations take place. This NNMs formulation can be named **the Shaw-Pierre approach**.

Basic results on NNMs are presented in the book [7-9] where quantitative and qualitative analyses of NNMs in conservative and non-conservative systems are considered.

An efficiency of the NNMs method is showed in some applied problems. In particular, NNMs in vibro-absorption problems are investigated in [10-12]. An essentially nonlinear oscillator, a snap-through truss with three equilibrium positions, and a vibro-impact oscillator are considered as absorbers. Construction and stability analysis of the localized and non-localized nonlinear normal modes are developed. If the localized mode is realized, the system energy is concentrated in the nonlinear absorber. This situation is the most appropriate to absorb vibrations of a linear subsystem. NNMs are used to analyze the cylindrical shell nonlinear dynamics [13]. Initial imperfections are taken into account. The Shaw-Pierre NNMs are used to analyze a 7-dof model for a double tracked road vehicle [14]. Nonlinear response of the suspension is taken into account.

The paper is organized as follow. In Section 2 the Kauderer-Rosenberg and Shaw-Pierre approaches are presented. In Section 3 an investigation of the NNMs in some pendulum systems is described. In Section 4 the Shaw-Pierre NNMs and the modified Rauscher method are used to construct the NNMs of forced vibrations in a one-disk rotor system.

2. Two approaches to the nonlinear normal modes

The Kauderer-Rosenberg approach of the NNMs is based on a construction of trajectories in the system configuration space. If some generalized coordinate is chosen as the independent one, for example, $x_1 \equiv x$, then the following functions defines the NNM:

$$x_i = x_i(x); \quad (i = 2,3,..., n) \tag{1}$$

For a finite-dof conservative system with the potential energy, $\Pi = \Pi(x)$, which is a positive definite analytical function, and the kinetic energy is transformed to the canonical form, the equations to obtain trajectories (modal lines) in the system configuration space can be written as [2, 7-9]

$$2x_i''(h-\Pi)/(1+\sum_{k=2}^{n} x_k'^2) + x_i'(-\Pi_x) = -\Pi_{x_i} \; (i = 2,3,...,n) \tag{2}$$

Here the prime means a differentiation by the variable x. These equations have singular points on the maximal equipotential surface $\Pi(x_1,...,x_n) = h$. An analytical continuation of the NNM trajectories to the surface is possible if the next boundary conditions are satisfied [2, 7-9], namely,

$$x_i'(X)[-\Pi_x(X, x_2(X)..., x_n(X))] = -\Pi_{x_i}(X, x_2(X)..., x_n(X)) \; (i = 2,3,..., n) \tag{3}$$

The relations (3) are natural boundary conditions for the variational principle in the Jacobi form. Here $x = X$, $x_i(X)$ are the trajectory return points lying on the maximal equipotential surface $\Pi(x_1,...,x_n) = h$. If the NNM (1) is obtained from the boundary problem (3)-(4), then the system under consideration is reduced to a single-dof nonlinear oscillator. So, the NNM is a two-parametric (by energy and phase of the motion) family of periodic solutions with smooth modal lines. To construct the NNM, a power series can be used [3,4].

Shaw and Pierre [5,6] reformulated the method of NNMs for a general class of quasilinear dissipative systems. Their analysis is based on the computation of invariant manifolds of motion on which the NNMs take place. One chooses a couple of new independent variables (*u, v*), where *u* is some *dominant generalized coordinate*, and *v is the corresponding generalized velocity*. By the Shaw-Pierre concept, the nonlinear normal mode is such a regime when all phase coordinates are univalent functions of the selected couple of variables. It is possible to obtain the solution in the power series by new independent variables u and v using a system of partial differential equations.

Generalization of the NNMs approach to non-autonomous and self-exited systems can be found in Refs. [7-9].

3. NNMs by Kauderer-Rosenberg in the pendulum dynamics

One considers free vibrations of the spring pendulum (Figure 1). Here m is a mass of the pendulum, l is length of the linear spring in the unstressed state. The dynamics of the system are described by two positional coordinates, ρ and φ.

One introduces a new variable $z = \rho - \rho_0$, where $\rho_0 = l + gm/c$ is a static extension of the spring in the equilibrium position. In the system two vibration modes can be selected: a) longitudinal vibration mode when $\varphi = 0$, $z = z(t)$; b) coupled vibration mode when $\varphi = \varphi(t)$, $z = z(t)$.

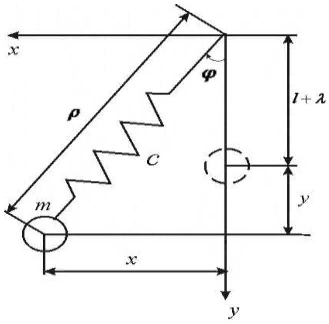


Figure 1. The model of the spring pendulum

The coupled vibration mode can be constructed in the form of the trajectory in the configuration space as $z = z(\varphi)$. Corresponding equation and boundary conditions are similar to equation (2) and (3). The NNM is determined as a power series by the independent variable φ. Coefficients of the series are obtained from algebraic equations. A numerical simulation needed for checking shows a good agreement with the obtained analytical solution.

The longitudinal vibration mode stability was studied previously [15] and results of such investigations are known. The corresponding linearized equation for variations in the horizontal direction can be reduced to the Mathieu equation. A more exact method for the stability analysis is based on reduction to the Hill equation. Analysis of stability of the coupled vibrations mode is made by the analytical-numerical criterion which is proposed in [16]. In this case current values of perturbations are compared with initial ones.

Free vibrations of the two-DOF system, which is shown in Figure 2, are considered too. The anchor spring is linear having a stiffness coefficient k. The linear oscillator of the mass m_1 is connected with the pendulum absorber, having the mass m_2; the length of the pendulum is equal to l. The system motions are determined by two generalized coordinates x and θ.

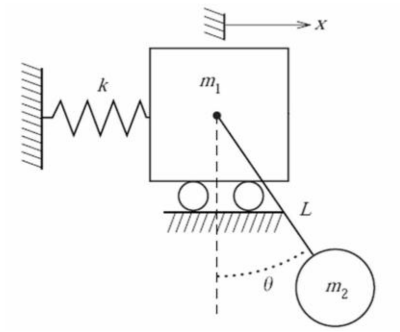


Figure 2. The mechanical system having the pendulum absorber

Two nonlinear vibration modes can be selected in this system: a) the coupled vibrations mode, $x = x(t), \theta = \theta(t)$, when vibration amplitudes of two generalized coordinates have the same order; b) the localized vibration mode, which is the most appropriate for absorption of the linear subsystem vibrations, when amplitudes of $\theta = \theta(t)$ are essentially larger than ones of $x = x(t)$.

The method of the NNMs construction presented in Section 2, is utilized here. The coupled vibration mode, $\theta = \theta(x)$, is found in a power series in x. The trajectory of the localized vibration mode is determined of the form: $x = x(\theta)$. The NNM trajectory is obtained as a power series in the variable θ. The near rectilinear trajectory of the localized mode is presented in Figure 3.

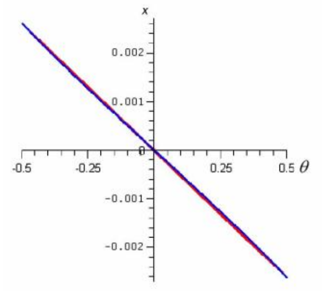


Figure 3. Trajectory of localized vibration mode in configuration space

Again the numerical simulation shows a very good agreement with the analytical solution.

The non-localized and localized NNMs trajectories are near rectilinear. Stability of the mode is determined by variations, which are orthogonal to the trajectories. The Mathieu and Hill equations are used in this investigation. The region of the localized mode instability is very narrow. So, the localized mode is very effective for absorption of elastic vibrations.

The forced NNMs in the system containing the pendulum absorber under the external periodic excitation are obtained too.

4. Nonlinear normal modes of forced vibrations in rotor systems

Different nonlinear effect must be taken into account in the analysis of the dynamical behaviour of rotor systems [17,18]. Note that in many publications mostly the simplest models, for example, the Jeffcott rotor, are considered due to a complexity of the rotor system dynamics. Here nonlinear forced vibrations of the rotor are considered. Gyroscopic effects, nonlinear flexible base, inertial forces in supports, an asymmetrical disposition of the disk in the shaft and internal resonances are all taken into account. Equations of motion of the one-disk unbalanced rotor with the linearly isotropic elastic shaft and nonlinear elastic bearings are the following [9]:

$$
\begin{aligned}
& m\ddot{x} + \rho_1\dot{x} + k_{11}\left(x - h_1x_2 - h_2x_1\right) + k_{12}\left(\theta_2 - \frac{x_2 - x_1}{l}\right) = \varepsilon\Omega^2 m\cos\Omega t \\
& m\ddot{y} + \rho_1\dot{y} + k_{11}\left(y - h_2y_1 - h_1y_2\right) - k_{12}\left(\theta_1 + \frac{y_2 - y_1}{l}\right) = \varepsilon\Omega^2 m\sin\Omega t \\
& I_e\ddot{\theta}_1 + \rho_2\dot{\theta}_1 + I_p\Omega\dot{\theta}_2 + k_{22}\left(\theta_1 + \frac{y_2 - y_1}{l}\right) - k_{12}\left(y - h_2y_1 - h_1y_2\right) = 0 \\
& I_e\ddot{\theta}_2 + \rho_2\dot{\theta}_2 - I_p\Omega\dot{\theta}_1 + k_{22}\left(\theta_2 - \frac{x_2 - x_1}{l}\right) + k_{12}\left(x - h_1x_2 - h_2x_1\right) = 0 \\
& m_1\ddot{x}_1 + \beta\dot{x}_1 + \left(\frac{k_{12}}{l} - h_2k_{11}\right)\left(x - h_1x_2 - h_2x_1\right) + \left(\frac{k_{22}}{l} - h_2k_{12}\right)\left(\theta_2 - \frac{x_2 - x_1}{l}\right) + c_x^{(1)}x_1 + c_x^{(2)}x_1^3 = 0 \\
& m_1\ddot{y}_1 + \beta\dot{y}_1 + \left(\frac{k_{12}}{l} - h_2k_{11}\right)\left(y - h_2y_1 - h_1y_2\right) + \left(h_2k_{12} - \frac{k_{22}}{l}\right)\left(\theta_1 + \frac{y_2 - y_1}{l}\right) + c_y^{(1)}y_1 + c_y^{(2)}y_1^3 = 0 \\
& m_1\ddot{x}_2 + \beta\dot{x}_2 + \left(-\frac{k_{12}}{l} - k_{11}h_1\right)\left(x - h_1x_2 - h_2x_1\right) + \left(-\frac{k_{22}}{l} - h_1k_{12}\right)\left(\theta_2 - \frac{x_2 - x_1}{l}\right) + k_x^{(1)}x_2 + k_x^{(2)}x_2^3 = 0 \\
& m_1\ddot{y}_2 + \beta\dot{y}_2 + \left(-\frac{k_{12}}{l} - h_2k_{11}\right)\left(y - h_2y_1 - h_1y_2\right) + \left(\frac{k_{22}}{l} + h_1k_{12}\right)\left(\theta_1 + \frac{y_2 - y_1}{l}\right) + k_y^{(1)}y_2 + k_y^{(2)}y_2^3 = 0
\end{aligned}
\tag{4}
$$

where l is the shaft length; l_1, l_2 are distances of the disk up to left and right supports, correspondently; $h_1 = l_1 / l; h_2 = l / l_2$; $c_x^{(1)}, c_y^{(1)}$ are coefficients which characterize linear terms in the left support restoring force; $k_x^{(1)}, k_y^{(1)}$ are similar coefficients for the right support; $c_x^{(2)}, c_y^{(2)}$ are coefficients which characterize cubic terms in the left support restoring force; $k_x^{(1)}, k_y^{(1)}$ are similar coefficients for the right support; β is a coefficient of damping in the supports; ρ_1, ρ_2 are coefficients of damping during the disk motion; m is the disk mass; ε is an eccentricity of the disk mass center.

One has an 8-dof nonlinear system, describing the displacements and rotations of the disc, and displacements of the nonlinear supports. The NNMs are constructed here in the forced rotor system having an internal resonance. This situation is always realized in the rotor dynamics with the isotropic-elastic shaft and supports. In a case of internal resonance it is possible to obtain a 2-dof nonlinear system for each vibration mode of the forced vibrations.

The Rauscher method was first proposed for the single-DOF system [19]. Generalization of the method to construct NNMs in general non-conservative systems is proposed in [20]. One considers the nonlinear dynamical system under the external harmonic excitation in normal coordinates.

Let $\bar{q} = \{q_1, q_2, ..., q_N\}^T$, $\bar{s} = \{s_1, s_2, ..s_N\}^T$ be the principal coordinates and corresponding velocities. It is assumed that two linearized frequencies ν_1 and ν_2 are close, and they are close to the external frequency, Ω, that is $\Omega \approx \nu_1 \approx \nu_2$. In this case two active coordinates, $q_{1,2}$, and two corresponding velocities, $s_{1,2}$, may be taken as independent master coordinates to construct the forced NNMs. One uses from the zero approximation a representation of the active coordinates as a Fourier series when the other coordinates are essentially smaller than the master ones:

$$
\begin{aligned}
q_1 &= A_1 cos(\Omega t) + B_1 sin(\Omega t) + A_2 cos(2\Omega t) + B_2 sin(2\Omega t) + A_3 cos(3\Omega t) + B_3 sin(3\Omega t) + ... \\
s_1 &= \Omega(B_1 cos(\Omega t) - A_1 sin(\Omega t) + 2B_2 cos(2\Omega t) - 2A_2 sin(2\Omega t) + 3B_3 cos(3\Omega t) - 3A_3 sin(3\Omega t) + ...) \\
q_2 &= ... \quad s_2 = ...
\end{aligned}
\tag{5}
$$

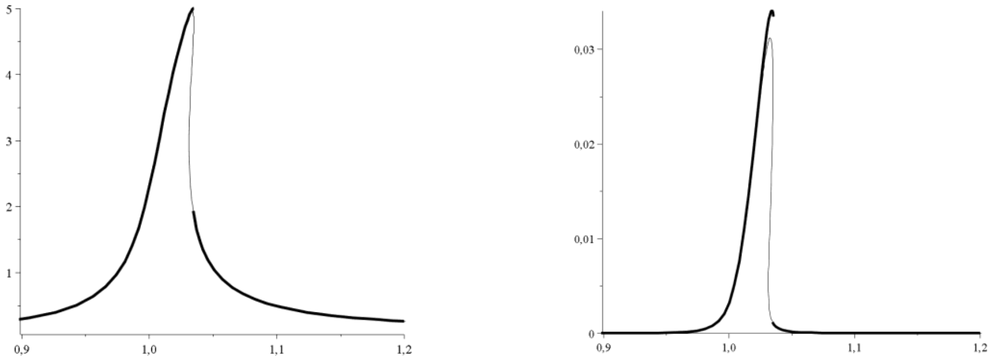
One has from here, after some trigonometric transformations, that

$$
cos(\Omega t) = \alpha_1 q_1 + \alpha_2 s_1 + \alpha_3 q_2 + \alpha_4 s_2 + \alpha_5 q_1^2 + \alpha_6 s_1^2 + ... \tag{6}
$$

When the coefficients of the expansions (6) are determined from algebraic equations, then an n-dof “pseudo-autonomous” system is obtained instead of the initial non-autonomous system. It corresponds to the principal idea of the Rausher method. In the obtained autonomous system, having the internal resonance, the NNMs can be constructed as functions of *four* independent coordinates q_1, s_1, q_2, s_2. Corresponding partial differential equations, similar to equations (8), are used. Solutions of these equations are obtained in the form of the power series by *four master coordinates*. It permits the reduction of the n-dof system to the two-dof one. Four active phase coordinates are obtained from this reduced system in a form of the Fourier series. The recurrent process is constructed, and the pointed out series of operations can be repeated some times to reach the necessary condition of exactness. So, the steady-state resonance regimes are constructed in the form of the NNMs.

The procedure of the NNMs construction is used in the rotor dynamics. Frequency responses near the first resonance are presented in Figure 4 for some values of the system parameters. Figures (a) and (b) represent frequency responses for principal coordinate q_1 (first and third harmonic of excitation frequency respectively). Trajectories of the resonance vibrations in the system configuration space are constructed too. Numerical simulation shows a good efficiency of the proposed approach.

a)



(b)

Figure 4. Frequency responses near the first resonance. Bold curves correspond to the NNM approach, and thin curves correspond to the calculations by the harmonic balance method. Here frequencies on the horizontal axis are dimensionless; all amplitudes are multiplied by a scaling coefficient of 1000.

5. Conclusions

Nonlinear normal modes (NNMs) are typical regimes realized in different conservative or near-conservative finite-dof systems. The method of nonlinear normal modes is one of the approaches for dimension reduction in nonlinear systems. The Kauderer-Rosenberg concept, when all positional coordinates are single-valued functions of some of them, is associated with trajectories in configuration space. The Shaw-Pierre concept is based on the computation of invariant manifolds of motion. In this case the NNMs can be obtained as single-valued functions of two selected phase coordinates. An efficiency of the NNMs theory is shown in some applied problems, in particular, in pendulum and rotor dynamics.

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