NONLINEAR NORMAL VIBRATION MODES AND THEIR APPLICATIONS

Yuri V. Mikhlin [1]

[1] National Technical University, Kharkov, Ukraine, muv@kpi.kharkov.ua

Abstract: Two conceptions of nonlinear normal vibration modes (NNMs) in conservative and near conservative systems are considered. Construction of the NNMs and applications in some mechanical problems are presented. Namely, the nonlinear vibro-absorption problem, the cylindrical shell nonlinear dynamics and the vehicle suspension nonlinear dynamics are analyzed.

Keywords: Nonlinear normal modes.

1. INTRODUCTION

Nonlinear normal vibration modes (NNMs) are a generalization of normal (or principal) vibrations in conservative linear systems In the normal vibration mode a finite dimensional system behaves like a conservative one having a single degree of freedom.

There is a remarkable example of the successful use of the linear normal vibrations to construct periodical solutions in nonlinear case. Namely, Lyapunov proved that nonlinear finite-dimensional systems with an analytical first integral (for example, the energy integral) allow a one-parameter family of periodic solutions which tend towards linear normal vibration modes as amplitudes tend to zero [1]. The obtained solutions possess properties of the normal vibration modes in linear systems.

Kauderer [2] became a forerunner in developing quantitative methods for analyzing NNMs in some two-DOF conservative system. Problems of existence of periodical solutions which are similar to the NNMs, in some classes of the Hamilton systems was first considered by Seifert [3]. Note that problems of the NNMs qualitative theory will not be discussed in this paper.

Then researches in this direction were continued by Rosenberg and his co-authors [4-6]. Rosenberg considered n- DOF conservative oscillators and defined NNMs as "vibrations in unison", i.e., synchronous periodic motions during which all coordinates of the system vibrate equiperiodically, reaching their maximum and minimum values at the same instant of time. He gave the first definition of the NNMs and selected broad classes of essentially nonlinear systems allowing nonlinear normal vibration modes with rectilinear trajectories in a configuration space, that is the direct generalization of the linear NNMs. For example, 'homogeneous systems' whose potential is an even homogeneous function of the variables belong to such a class. The first conception of the NNMs can be named **the Kauderer-Rosenberg conception**.

In general, the NNM trajectories in a configurational space are curvilinear. For some particular cases curvilinear trajectories were studied in [6] and [7]. The power series method was proposed to construct the trajectories by Manevich and Mikhlin [8-10]. In [11] nonlinear normal mode localization is studied in non-linear systems with impact nonlinearities. In [12] for an oscillator with weak coupling stiffness, both localized and non-localized modes are detected using an asymptotic methodology. In [13] the NNMs and global dynamics of nonlinear systems are analyzed by using the Poincaré map. In [14] Padé approximations are used for an analysis of the NNMs with large amplitudes.

Show and Pierre [15,16] reformulated the concept of NNMs for a general class of nonlinear discrete conservative, or non-conservative systems. This analysis is based on the computation of invariant manifolds of motion on which the NNM oscillations take place. Note that similar construction was used too by Lyapunov [1]. This second conception of the NNMs can be named **the Lyapunov-Shaw-Pierre conception**.

Note that publications on the generalization of NNMs to non-conservative systems are not numerous. It should be noted that periodic solutions in non-autonomous systems close to Lyapunov systems were thoroughly investigated by Malkin [17]. The Rauscher method was first used for analyses of normal vibrations in non-autonomous systems in [18]. One assumes that the generative autonomous system is homogeneous. NNMs in general finite-dimensional non-autonomous systems close to conservative ones allowing similar NNMs were considered in [19]. Rauscher's ideas and the power-series method for trajectories in a configurational space are used in the construction of resonance solutions. Some other results on the NNMs in non-autonomous systems was presented in [20-22]. NNMs in near-conservative self-excited systems were analyzed in [23]. NNMs of parametric vibrations were described in [24].

Basic results on NNMs are presented in the book by Vakakis et al. [25] which describes quantitative and qualitative analyses of NNMs in conservative and non-autonomous systems, including localized modes, an analysis of stability, and an investigation of NNMs in distributed systems.

Symmetry properties of the nonlinear system and a problem of the NNMs existence was investigated in publications [25,26]. Some generalization of the NNMs conception on the basis of the group theory was presented in [27]. NNMs in systems with non-smooth characteristics were analyzed in papers [11,28]. The so-called saw-tooth

time transformations were used. In [29] was shown that the closed trajectories of the NNMs exist in a case of gyroscopic forces in the system under consideration. In [30] the NNMs are presented by using functions of some selected scalar phase.

Generalization of the NNMs conceptions to continuous systems was made in papers [31-33]. Two different principal generalizations for the continuous systems can named the King-Vakakis approach [31] and the Show-Pierre approach [32]. These interesting approaches will not be discussed in the presented paper.

Publications on the NNMs stability problems are not presented here.

The nonlinear normal modes theory have been used last years to solve different applied problems In particular, it is possible to mention the nonlinear vibro-absorption problem [34-39], the shell dynamics [40], the shallow arcs dynamics [41] et al.

The paper is organized as follow. In Section 2 the Kauderer-Rosenberg conception of the nonlinear normal modes is presented. A construction of equations and boundary conditions in the finite-DOF conservative systems to obtain the NNMs trajectories (modal lines) in configurational space, is shown. Some properties of the NNMs are presented. Construction of the NNMs trajectories in power series is described in this Section. In Section 3 a generalization of the Kaudered-Rosenberg conception to non-conservative systems is considered. The principal idea of the Shaw-Pierre conception for the nonlinear normal modes is described in Section 4. Then some applications of the nonlinear normal modes theory are presented. In Section 5 it is shown that the snap-through truss can be used as effective vibration absorber. Non-localized and localized NNMs are selected and constructed. Their stability is analyzed too. It is shown that the stable localized NNM, when the system energy is concentrated in the nonlinear absorber, is very effective for the absorption. In Section 6 the NNMs theory is used investigate the nonlinear dynamics of cylindrical shells with initial imperfections, including the shells dynamics in supersonic flow. In Section 7 a nonlinear 7-DOF model of the vertical and axial vehicle dynamics is considered. The NNMs approach is used both for smooth, and for non-smooth characteristics of the car suspension.

2. THE CONCEPTION OF THE NNMs BY KAUDERER AND ROSENBERG

2.1. Trajectories of the NNMs in the configuration space

The normal vibrations of the conservative systems can be presented by trajectories in the configuration space. One considers the finite-DOF conservative system having the single trivial position of static equilibrium. Equation of motion are the following:

$$m_i \ddot{x}_i + \Pi_{x_i} = 0 \qquad (1)$$

where $\Pi = \Pi(x)$ is the system potential energy, which is assumed as the positive definite analytical function, and $\Pi_z \equiv \dfrac{\partial \Pi}{\partial z}$. One introduces the next change of variables,

$\sqrt{m_i}\, x_i \to x_i$. Then the kinetic function is reduced to a sum of squares of coordinates and the energy integral of the system can be wrote as

$$\frac{1}{2}\sum_{k=1}^{n} \dot{x}_k^2 + \Pi(x_1, x_2, ..., x_n) = h \qquad (2)$$

where h being the system energy. Note, that all motions here are bounded by a closed maximum equipotential surface $\Pi(x_1, ..., x_n) = h$. On this surface one has $\dot{x}_1 = \dot{x}_2 = ... = \dot{x}_n = 0$.

Equations to obtain the trajectories of motions in the system configurational space can be obtained as the Euler equations for the Jacobi variation principle of the form

$$\delta S = \delta \int_{P_1}^{P_2} \sqrt{2(h - \Pi)}\, ds = 0, \qquad (3)$$

where $ds^2 = \sum_{k=1}^{n} dx_k^2$, or

$$\delta S = \delta \int_{\alpha(P_1)}^{\alpha(P_2)} \sqrt{2(h - \Pi)} \sqrt{\sum_{k=1}^{n} (x_k')^2}\, d\alpha = 0, \text{ where } \alpha$$

is the arbitrary parameter, and the prime means a differentiation by the parameter. If some generalized coordinate is chosen as the varied parameter, for example, $x_1 \equiv x$, then the following functions must be obtained:

$$x_i = p_i(x); \;\; (i = 2,3,...,n) \qquad (4)$$

The last relations define the NNMs by Kauderer-Rosenberg. These are so-called "vibrations in unison", which describe some synchronous periodic motions during which all positional coordinates of the system vibrate equiperiodically. To write the equations to obtain the functions (4), it can use the next relations, when the new independent variable x is introduced instead of t:

$$\frac{d}{dt} = \dot{x}\frac{d}{dx}; \; \frac{d^2}{dt^2} = \dot{x}^2 \frac{d^2}{dx^2} + \ddot{x}\frac{d}{dx}. \qquad (5)$$

Excluding $\dot{x}^2$ from the energy integral, one has as a result the following relation:

$$\dot{x}^2 = 2(h - \Pi)/(1 + \sum_{k=2}^{n} x_k'^2) \qquad (6)$$

Here and later the prime means a differentiation by variable x.

Then we can derive equations to obtain trajectories (modal lines) in the configurational space directly from the equations (1) using the relations (5) and (6). One has the following equations:

$$2x_i'' \frac{h - \Pi}{1 + \sum\limits_{k=2}^{n} x_k'^2} + x_i'(-\Pi_x) = -\Pi_{x_i} \ (i = 2,3,...,n) \ (7)$$

Although a number of the equations (7) is smaller by a unit than a number of the equations of motion (1), the equations (7) are non-autonomous and nonlinear, even for the linear conservative systems. Moreover, these equations have singular points on the maximal equipotential surface $\Pi(x_1,...,x_n) = h$. In fact, these equations are convenient to construct rectilinear, or nearly rectilinear, modal lines of the NNMs [4,5,8-10,25].

One considers trajectories which reach the maximum equipotential surface $\Pi(x_1,...,x_n) = h$. An analytical continuation of these trajectories to the surface is possible if the next boundary conditions are satisfied [4,5,8-10,25], namely,

$$x_i'(X)[-\Pi_x(X,x_2(X)...,x_n(X))] =$$
$$-\Pi_{x_i}(X,x_2(X)...,x_n(X))$$
;
$$(i = 2,3,...,n) \tag{8}$$

Here $x = X$, $x_i(X)$ are the trajectory return points (amplitude values) lying on the maximal equipotential surface $\Pi(x_1,...,x_n) = h$. The conditions (8) define an orthogonality of the NNM trajectory to the maximum equipotential surface. On the other hand, these conditions are the natural boundary conditions for the Jacobi variation principle (3).

Let's the trajectory (5) is obtained from the equations (7) and the boundary conditions (8), and the functions $p_i(x)$ are single-valued and analytical. Then the law of motion in time t can be found by using the equation

$$\ddot{x} + \Pi_x(x_1, p_2(x),..., p_n(x)) = 0 \tag{9}$$

If this equation is presented of the form $\ddot{x} + V'(x) = 0$, the periodic solution $x(t)$ can be found as inversion of the integral

$$t + \varphi = \frac{1}{\sqrt{2}} \int_X^x \frac{d\xi}{\sqrt{h - V(\xi)}} \tag{10}$$

So, the nonlinear normal modes are two-parametric (by energy and phase of the motion) family of periodic solutions with smooth trajectories in the configurational space.

Note that the energy and amplitudes of the obtained single-DOF nonlinear system are connected by the following relation:

$$h = V(X) \tag{11}$$

Condition of closure of all equipotential surfaces for different values of the energy h, and an absent of any equilibrium positions, excepting the trivial one, $x_i = 0$ $(i = 1,2,...,n)$, guarantee a solvability of the equation (11) with respect to amplitude values X for a given value of the energy h [8,10,25]. The power series expansion of the potential energy $V(x)$ begins from the terms of the even power, so two amplitude values X_i $(i = 1,2)$ correspond to each fixed value of the energy h. If the potential energy contains only even powers terms, one has $X_1 = -X_2$.
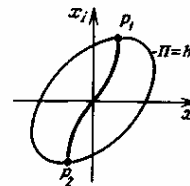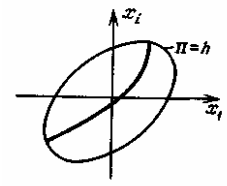

Fig.1,a   Fig. 1,b

**Fig. 1. The curvilinear NNM trajectory in conservative systems
(Fig. 1,a. The trajectory meet the origin; Fig 1,b. The trajectory does not meet the origin)**

2.2. *Some properties of the nonlinear normal modes*

R.Rosenberg [4,5] selected broad classes of essentially nonlinear conservative systems allowing NNMs with rectilinear modal lines, that is,

$$x_i = k_i x; \ (k_i = const), \tag{12}$$

where $k_i -$ are so-called modal constants.

For instance, homogeneous systems which potential energy is the even homogeneous function of the positional variables belong to such a class.

Rosenberg showed that rectilinear trajectories of the NNMs, $x_i = k_i x; \ (k_i = const)$, intersect all equipotential surfaces orthogonally [4,5]. In fact, in this case the equations (2.7) can be reduced to the next form:
$$x_i'(-\Pi_x(x,x_2,...,x_n)) = -\Pi_{x_i}(x,x_2,...,x_n);$$

$(i = 2,3,...,n)$, and those are conditions of orthogonality of the rectilinear trajectories with the family of equipotential surfaces $\Pi(x_1,x_2...,x_n) = const$.

It is very interesting to note that the number of NNMs in the nonlinear case can exceed the number of degrees of freedom. This remarkable property has no analogy in general linear systems, excepting some degenerate cases. As an example, one considers the next system:

$$\ddot{x}_1 + x_1 + x_1^3 + \gamma(x_1 - x_2)^3 = 0;$$
$$\ddot{x}_2 + x_2 + x_2^3 + \gamma(x_2 - x_1)^3 = 0. \qquad (13)$$

The rectilinear NNMs of the form $x_2(t) = cx_1(t)$, are determined by the following algebraic equation for the modal constant c:

$$c\left(1 - c^2\right) + \gamma\left(1 - c\right)^3(1 + c) = 0. \qquad (14)$$

One has from here that the equation (14) always has two solutions, $c_{1,2} = \pm 1$, which corresponds to the in-phase and anti-phase NNMs. But for $\gamma < 0.25$ the equation (14) possesses two other real roots,

$$c_{3,4} = \frac{1}{2}\left[-\frac{1}{\gamma} + 2 \pm \sqrt{\frac{1}{\gamma}\left(\frac{1}{\gamma} - 4\right)}\right]. \qquad \text{Corresponding}$$

NNMs bifurcate from the anti-phase mode at the critical point $\gamma = 0.25$, when the anti-phase mode becomes unstable. So, if $\gamma < 0.25$, four NNMs exist in the nonlinear system. In a limit $\gamma \rightarrow 0$, two additional NNMs tend to separate motions of partial oscillators when a connection is absent.

2.3. *Construction of curvilinear trajectories of the NNMs.*

In [8-10,25] a construction of the NNMs trajectories in power series was presented.

One considers the following dynamical system:

$$\ddot{q}_i + \Pi_{q_i}^{(0)}(q_1,...,q_n) + \varepsilon \Pi_{q_i}^{(1)}(q_1,...,q_n) = 0; (i = \overrightarrow{1,n}), \quad (15)$$

where $\varepsilon << 1$; $\Pi^{(0)} + \varepsilon\Pi^{(1)} -$ is the system potential energy.

One assumes that the unperturbed system (for $\varepsilon = 0$) allows the rectilinear modal lines as $q_i = k_i q; \ (q \equiv q_1; 2 \le i \le n)$. Some of the rectilinear modal lines is chosen as the generative one. By rotation of coordinates, $q_i \rightarrow x_i$, the solution can be presented as

$$x_i = 0; (i = 2,3,...,n); \ x \equiv x_1(t) \qquad (16)$$

The system (15) can be rewritten as

$$\ddot{x}_i + \tilde{\Pi}_{x_i}^{(0)}(x_1,...,x_n) + \varepsilon\tilde{\Pi}_{x_i}^{(1)}(x_1,...,x_n) = 0; i = \overrightarrow{1,n}. \quad (17)$$

To construct the NNM close to the solution (16) it can use the following power series:

$$x_i = \sum_{k=1} \varepsilon^k x_{ik}(x); \ x \equiv x_1 \qquad (18)$$

Here the functions $x_{ik}(x)$ must be presented as power series by x:

$$x_{ik} = \sum_{l=1} a_{ik}^{(l)} x^l \qquad (19)$$

A determination of the power series (19) coefficients from the equations for the trajectories in configurational space and the corresponding boundary conditions is described in [8-10,25]. Detailed presentation of this procedure is not presented here. Conditions of solvability and a convergence of the power series are discussed in these publications too.

In a case when a potential energy of the unperturbed system (17), $\tilde{\Pi}^{(0)}(x, x_2,..., x_n)$, is the even homogeneous function with power $r + 1$ on all generalized coordinates, the conditions of existence of the unique NNM close to the rectilinear NNM (16) are the following [8-10,25]:

$$K_p \ne 0, \qquad (20)$$

where $K_p = Det(Q); \ p = 0,1,...; \ Q = \|q_{ij}\|;$

$$q_{ij} = \delta_{ij}\left|p(p-1)2\tilde{\Pi}^{(0)}(1,0,...0) + p\tilde{\Pi}_x^{(0)}(1,0,...0) - \tilde{\Pi}_{x_ix_i}^{(0)}(1,0,...0)\right|,$$

(δ_{ij} is the Kronecker symbol). If the unperturbed system is linear, the conditions (20) mean that internal resonances in the linear system is eliminated from consideration.

If the NNM trajectory is constructed as power series by some selected positional coordinate, an analytical continuation of obtained local expansions for the large vibration amplitudes can be made by using the rational diagonal Pade' approximants [10,14].

Concrete calculations realized for two DOF nonlinear systems show a very good accuracy of the proposed approach.

3. NNMs IN NEAR-CONSERVATIVE SYSTEMS

The perturbation methodology can be utilized to the analysis of NNMs in broad classes of finite-dimensional non-autonomous and self-excited systems close to conservative ones.

One considers the following near-conservative system:

$$\ddot{x}_i + \Pi_{x_i}(x_1,x_2,...x_n) + \varepsilon f_i(x_1,\dot{x}_1,x_2,\dot{x}_2,...x_n,\dot{x}_n,t) = 0; \left(i = \overrightarrow{1,n}\right). \quad (21)$$

Here ε is a small parameter; Π is a potential energy of the unperturbed conservative system; the functions f_i may be periodical with respect to time, or may contain terms which guarantee an appearance of limit cycles. So, this system may involve friction of any physical nature, such as viscous, dry, or turbulent.

It is considered such vibration modes when all positional coordinates of the finite-dimensional non-conservative system are linked. Along trajectories of these modes, the system behaves like a single-DOF conservative one. The periodic solutions could be called *Nonlinear normal modes (NNMs) of the non-conservative nonlinear system.*

Let us find a solution that all phase coordinates are defined as single-valued and analytical functions of some selected coordinate $x \equiv x_1$. Besides, it is assumed that within a semi-period of the periodic solution one can express t as a single-values function of the displacement x. This idea was first introduced by Rausher in 1938 [42]. It means that along the NNM trajectory the non-conservative system behaves like to some *pseudo-autonomous system* [10,19,25].

So, one presents:

$$x_i = x_i(x, \varepsilon); \dot{x} = \dot{x}(x, \varepsilon); \dot{x}_i = \dot{x}_i(x, \varepsilon); t = t(x, \varepsilon) \quad (22)$$

Then, eliminating a time, introducing the new independent variable x, and using the relations $\dfrac{d}{dt} = \dot{x}\dfrac{d}{dx}$; $\dfrac{d^2}{dt^2} = \dot{x}^2 \dfrac{d^2}{dx^2} + \ddot{x}\dfrac{d}{dx}$, one obtains as a result, the following equations governing trajectories of the NNMs in the system configurational space [10,19,25]:

$$2x_i''(\dot{x}(x))^2 + x_i'\begin{bmatrix} -\Pi_x(x,x_2(x),...x_n(x)) + \\ g f(x,\dot{x}(x),x_2(x),...t(x)) \end{bmatrix} +$$
$$+ \Pi_{x_i}(x,x_2(x),...x_n(x)) + g f_i(x,\dot{x}(x),x_2(x),\dot{x}_2(x),...t(x)) = 0;$$
$$(i = \overline{2,n}) \quad (23)$$

Here the prime denotes differentiation with respect to x; $f(...) \equiv f_1(...)$.

The equations (23) have singularities in the trajectory return points where all velocities are equal to zero, $x = X_j; (j = 1,2)$. As for conservative systems, these singularities are removable, if the following additional boundary conditions are satisfied:

$$\{x_i'[-\Pi_x(x,x_2(x),...x_n(x)) + g f(x,\dot{x}(x),x_2(x),\dot{x}_2(x),...t(x))] +$$
$$+ \Pi_{x_i}(x,x_2(x),...) + g f_i(x,\dot{x}(x),x_2(x),\dot{x}_2(x),...t(x))\}\Big|_{x=X_j} = 0;$$
$$(i = \overline{2,n}; j = 1,2) \quad (24)$$

Note that these equations (23) and boundary conditions (24) are similar in form to ones previously considered in a case of conservative systems. It allows a univalent determination of the trajectory $x_i(x)$ close to the rectilinear NNM, $x_{i0} = k_i x \ (k_i = const.)$, of the generative conservative system ($\varepsilon = 0$). Moreover, the NNM trajectory can be determined in form of power series by ε and x. If the trajectory is found, then the equation of motion along the trajectory is the following:

$$\ddot{x} + \Pi_x(x,x_2(x),...x_n(x)) + g f(x,\dot{x}(x),x_2(x),\dot{x}_2(x),...t(x)) = 0 \quad (25)$$

So, along the NNM trajectory the non-conservative systems behaves like to the single-DOF conservative system which motion is determined by the function $x(t)$. We can write the energy integral for the single-DOF conservative system:

$$\frac{\dot{x}^2}{2} + \Pi(x,x_2(x),...,x_n(x)) + \varepsilon F(x,\varepsilon) = h(\varepsilon) \quad (26)$$

Note that the function $F(x,\varepsilon)$ is defined in process of the solution construction, and it is not, in general, analytical by x. One stresses that the condition (26) make sense for the non-conservative only along the closed trajectory of the NNM.

One has from the equation (26) the following quadrature:

$$t(x,\varepsilon) + \varphi = \frac{1}{\sqrt{2}} \int_{x_0}^{x} \frac{d\xi}{\sqrt{h(\varepsilon) - \Pi(\xi,x_2(\xi),...,x_n(\xi)) - \varepsilon F(\xi,\varepsilon)}}, \quad (27)$$

where φ is the arbitrary phase of the solution. From the relation (27) the variable t can be obtained as a function of the generalized coordinate x, which corresponds to the principal idea of the Rauscher method. All presented relations permit to construct the iterative procedure of the NNM construction, using as zero approximation the chosen NNM of the generative conservative system. Details of this construction are discussed in publications [10,19,23,25].

In a case of the non-autonomous system a procedure of determination of the steady-state resonance motions in form of the NNMs, must be completed by the next periodicity condition:

$$T + \varphi = \frac{1}{\sqrt{2}} \oint \frac{d\xi}{\sqrt{h(\varepsilon) - \Pi(\xi,x_2(\xi),...,x_n(\xi)) - \varepsilon F(\xi,\varepsilon)}} \quad (28)$$

where T is a period of the external excitation; the integration conducts during a complete period of steady-state motion.

In a case of the self-excited system one has the condition that the work of all forces over the period is equal to zero [23]:

$$\oint f(\xi,\dot{\xi}(\xi),x_2(\xi),\dot{x}_2(\xi),...,x_n(\xi),\dot{x}_n(\xi))d\xi = 0 \quad (29)$$

Both the periodicity condition (28) and the condition (29) mean that a loss of energy on the average over the period of the periodic solutions under consideration, is absent. So, these conditions can be named *potentiality conditions*.

4. THE NNMs CONCEPTION BY SHOW AND PIERRE.

In [15,16] Shaw and Pierre reformulated the NNMs concept for a general class of nonlinear discrete oscillators. The analysis is based on the computation of invariant manifolds of motion on which the NNMs take place. Some mechanical applications of this approach can be found in publications by the Shaw and Pierre.

To use this approach the mechanical system must be presented of the next standard form:

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = f(x,y). \end{cases} \qquad (30)$$

where $x = \{ x_1 \ \ldots \ x_N \}^T$ is a vector of the generalized coordinates, $y = \{ y_1 \ \ldots \ y_N \}^T$ is a vector of the generalized velocities, and $f = \{ f_1 \ \ldots \ f_N \}^T$ is a vector of the forces. One chooses a couple of new independent variables (u, v), where u is some **dominant generalized coordinate,** and **v is the corresponding generalized velocity**. By the Shaw-Pierre approach, the nonlinear normal mode is such regime when all generalized coordinates and velocities are univalent functions of the selected couple of variables. Choosing as the selected couple of variables, the coordinate and velocity with the index 1, one writes the nonlinear normal mode as

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_N \\ y_N \end{pmatrix} = \begin{pmatrix} u \\ v \\ X_2(u,v) \\ Y_2(u,v) \\ \vdots \\ X_N(u,v) \\ X_N(u,v) \end{pmatrix}. \qquad (31)$$

Computing derivatives of all variables in the relations (312), and taking into account that $u = u(t)$ and $v = v(t)$, then substituting the obtained expressions to the system (30), one has the following system of partial derivation equations:

$$\begin{aligned} \frac{\partial X_i}{\partial u} v + \frac{\partial X_i}{\partial v} f_1(x,y) &= Y_i(u,v) \\ \frac{\partial Y_i}{\partial u} v + \frac{\partial Y_i}{\partial v} f_1(x,y) &= f_i(x,y) \quad i = \overline{1..N}. \end{aligned} \qquad (32)$$

One presents the system solution in the form of the power series by new independent variables u and v:

$$\begin{aligned} x_i = X_i(u,v) &= a_{1i}u + a_{2i}v + a_{3i}u^2 + a_{4i}uv + a_{5i}v^2 + \ldots, \\ y_i = Y_i(u,v) &= b_{1i}u + b_{2i}v + b_{3i}u^2 + b_{4i}uv + b_{5i}v^2 + \ldots. \end{aligned} \qquad (33)$$

The series (33) are introduced to the equations (32), then coefficients in terms of the same degree by independent variables, are equated. So, a system of recurrent algebraic equations can be written. Coefficients of the series (33) are determined from these equations, and, as a result, the corresponding nonlinear normal mode is obtained.

5. SNAP-THROUGH TRUSS AS A VIBRATION ABSORBER

Numerous scientific publications contain a description and analysis of different devices for the vibration absorption. It is not appropriate to refer here even the most principal, pioneer works by Frahm (1909), Den Hartog (1934), Roberson (1952) et al. One refers here some more late publications. In particular, Haxton and Barr [43] considered the absorber in the form of a beam, which is attached to the mass-spring system. Natsiavas [44,45] used the nonlinear oscillator to reduce the forced oscillations, or vibrations of the self-excited system. The linear and nonlinear absorbers general theory is presented in the handbook [46]**.** Haddow and Shaw [47] studied experimentally the rotating machinery with the centrifugal pendulum absorber. Problems of the energy localization and a use of it to absorb the vibration energy are studied in [48, 49]. Impact systems can be used to absorb oscillations too [50,51]. The inverted pendulum with motions limiting stops was used as a vibrations absorber by Shaw J, Shaw S.W.[52]. Analysis of the nonlinear normal modes in system, which contains the main linear subsystem and the essentially nonlinear oscillator as absorber, was made in [53]. The well known in mechanics snap-through truss, first studied my Mises (1923), was used in vibrations insulation systems with the quasi-zero stiffness [54]. A possibility of the free or forced oscillations absorption by using of the snap-through truss was considered by Avramov and Mikhlin [36-38].

5.1. *Equations of motion.*

A single-DOF linear oscillator is chosen as the most simple model of some elastic subsystem. The nonlinear absorber with three equilibrium positions (this is the snap-through truss) is attached to this oscillator. The dynamics of the system is studied by the nonlinear normal vibration modes approach. If the localized mode is realized, the system energy is concentrated in the nonlinear absorber. This situation is the most appropriate to absorb vibrations. Fig. 2 shows the system under consideration. Equations of motions have the following form:

$$M_* \ddot{u} + ku + c_* \left[u - \cos\varphi + \left\{ 1 + \frac{w^2}{(\cos\varphi - u)^2} \right\}^{-1/2} \right] = 0;$$

$$m_* \ddot{w} + c_* w \left[2 - \left\{ (\cos\varphi - u)^2 + w^2 \right\}^{-1/2} - \left\{ \cos^2\varphi + w^2 \right\}^{-1/2} \right] = 0;$$

$$c_* = cL^2; k = c_1 L^2; M_* = ML^2; m_* = mL^2,$$

$$(34)$$

where φ is the angle, which is defined the equilibrium position; L is a length of the undistorted spring; c is a spring stiffness of the truss; c_1 is a stiffness of the main

elastic system. Variables (u,w) of the equations (34) are dimensionless. The following relations connect (u,w) to system generalized coordinates (U,W): $u = \dfrac{U}{L}; w = \dfrac{W}{L}$.
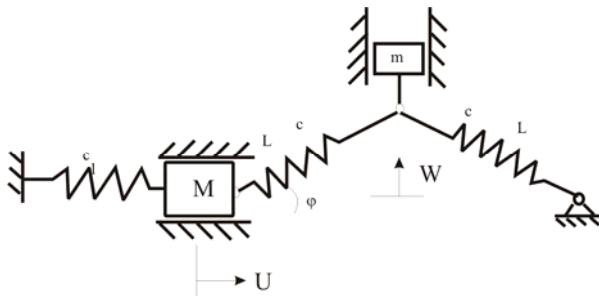

Fig.2. The snap-through truss as an absorber of elastic vibrations.

Let us introduce the dimensionless time: $t = (M / c_1)^{1/2} \tau$. The new variable is introduced: $u_1 = u + \dfrac{\gamma(1-c)}{1+\gamma}$, where $\gamma = \dfrac{c}{c_1}; \mu = \dfrac{m}{M}$. If the vibration absorption takes place, then $u_1 \ll w$. By assumption, the mass and stiffness of the truss are significantly smaller than the corresponding parameters of the elastic system. Therefore, the following relations are introduced: $\mu = \varepsilon\bar{\mu}; \gamma = \varepsilon\bar{\gamma}; \varepsilon \prec\prec 1$. The system (34) is rewritten with respect to (u_1, w) and the Taylor-series expansions are performed. The truss is shallow, so, remaining the terms up to the third order, the following system of the differential equations is derived:

$$\ddot{u}_1 + (1 + \varepsilon\bar{\gamma})u_1 - \frac{\varepsilon\bar{\gamma}}{\rho^3} u_1 w^2 - \frac{\varepsilon\bar{\gamma}}{2\rho^2} w^2 = 0;$$

$$\mu\ddot{w} - \bar{\gamma}\alpha^2 w - \frac{\bar{\gamma}}{\rho^2} w u_1 + \frac{\bar{\gamma}\beta^2}{2} w^3 = 0, \qquad (35)$$

where $\rho = \dfrac{\gamma + c}{1 + \gamma}; \alpha^2 = \dfrac{1}{\rho} + \dfrac{1}{c} - 2; \beta^2 = \dfrac{1}{\rho^3} + \dfrac{1}{c^3}$.

5.2. *Periodic motions with small amplitudes*

Nonlinear vibration modes of the snap-through truss were analysed in [36]. As the localyzed NNM, as well the non-localyzed NNM can be selected.

The localized periodic motions are presented of the form $u_1 = \varepsilon\bar{u}_1(w_1)$, where the following change of variables was made: $w - \sin\varphi = \varepsilon w_1$, and the following power series are used: $\bar{u}_1 = b_0 + b_1 w_1 + b_2 w_1^2 + ...$ Then the

equations to obtain the NNM trajectory and the corresponding boundary conditions are used. As a result, the localized NNM has been derived: $u_1 = \varepsilon b_1 w + O(\varepsilon^2)$. Thus the obtained trajectory in configuration space $(w_1, u_1) \in R^2$ is a near-straight line close to y-axis. The obtained solution accuracy is illustrated by the Runge-Kutta procedure for the system (35) with the following parameters: $\varphi = 0.15; \mu = \gamma = \varepsilon = 0.01$. The initial conditions are chosen from the analytical construction, that is $\dot{u}_1(0) = \dot{w}(0) = 0; w(0) = W_*^{(\max)}; \quad u_1(0) = \varepsilon b_1 W_*^{(\max)}$.

Fig.3 shows a very good agreement of the analytical solution and the numerical calculations, which are presented in the configuration space.
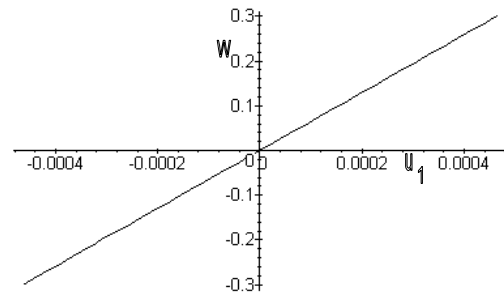

Fig.3 The localized NNM with small amplitudes.

The non-localized NNM has the following form: $u_1 = kw_1 + \varepsilon u_2(w_1)$, and the power series are used, that is $u_2 = c_0 + c_1 w_1 + c_2 w_1^2 + ...$ One has after calculations:

$$c_0 = -\frac{hB}{gs^2(s^2p^2 - 2)}; c_1 = \bar{\mu}\left(\frac{1}{g} - \frac{c}{s}\right); c_2 = \frac{p^2 gB}{2(s^2 p^2 - 2)}; c_3 = c_4 = 0.$$

The system (35) is integrated numerically. The initial conditions corresponding to the NNM, are the following: $u_1(0) = kW_1^{(\max)} + \varepsilon u_2(W_1^{(\max)}); w_1(0) = W_1^{(\max)};$ $\dot{u}_1(0) = \dot{w}_1(0) = 0$. The numerical simulation shows a good accuracy of the analytical results.

5.3. *Localized periodic motions with large amplitudes*

Periodic motions of the system (35) with large amplitudes are studied by using the NNMs approach too. The system (35) can be written in the following form:

$$\ddot{u}_1 + \frac{\partial\Pi}{\partial u_1} = 0; \mu\ddot{w} + \frac{\partial\Pi}{\partial w} = 0; \qquad (36)$$

where

$$\Pi = (1 + \varepsilon\bar{\gamma})\frac{u_1^2}{2} - \frac{\varepsilon\bar{\gamma}u_1^2}{2\rho^3} - \frac{\varepsilon\bar{\gamma}w^2 u_1}{2\rho^2} - \frac{\varepsilon\bar{\gamma}\alpha^2 w^2}{2} + \frac{\varepsilon\bar{\gamma}\beta^2 w^4}{8},$$

is the system potential energy. Let us determine the system (5.3) periodic motions in the form $u_1=u_1(w)$. The equation of trajectories in the configuration space can be wrote as

$$u_1''(w)\frac{2(h - \Pi)}{u_1'^2 + \mu} - \frac{1}{\mu}\Pi'_w u_1' = -\Pi'_{u_1}. \qquad (37)$$

The solution of the equation (37) is presented in the following form:

$$u_1 = \varepsilon\bar{u}_1(w); \bar{u}_1(w) = a_0 + a_1 w + a_2 w^2 + ... \qquad (38)$$

where $a_0, a_1, ...$ are unknown coefficients. Series (38) is substituted into (37) and matching of respective powers of w is carried out. Being restricted oneself to orders w^0, w^1, w^2, three linear algebraic equations with respect to five variables $a_0, ...a_4$ are derived. Two boundary conditions at the maximal equipotential surface $\Pi=h$ give us two additional algebraic equations:

$$\left\{ \frac{1}{\mu}\Pi'_w u_1' - \Pi'_{u_1} \right\}\Bigg|_{w = \pm W_*}, \qquad (39)$$

where W_* is an amplitude of the NNM. The equation for W_* determination is obtained from the energy integral when kinetic energy T=0, $\Pi|_{w = \pm W_*} = h$. We derive two equations from the boundary conditions (39). Solving the system of five linear algebraic equations with respect to $a_0, ...a_4$, one has:

$$a_1 = a_3 = 0; a_0 = -\frac{4\bar{h}}{\bar{\mu}}a_2;$$

$$\tilde{a}_2 = \left\{ \left(2 - \frac{(\bar{\mu} + 4\bar{\gamma}\alpha^2)W_*^2}{12\bar{h}} \right)\left(4\bar{\gamma}\alpha^2 - 2\bar{\gamma}\beta^2 W_*^2 + \bar{\mu} \right) + \bar{\gamma}\beta^2 W_*^2 \right\}^{-1} \times$$

$$\left\{ 2\bar{\mu} - \frac{\bar{\mu}W_*^2}{12\bar{h}}\left(4\bar{\gamma}\alpha^2 - 2\bar{\gamma}\beta^2 W_*^2 + \bar{\mu} \right) \right\};$$

$$\tilde{a}_4 = \frac{\bar{\mu}}{24\bar{h}} - \frac{\bar{\mu} + 4\bar{\gamma}\alpha^2}{24\bar{h}}\tilde{a}_2; a_j = \tilde{a}_j\frac{\bar{\gamma}}{2\rho^2}; j = 2,4. \qquad (40)$$

So, the NNM trajectory is obtained in the form (38). The following values of the parameters are taken for numerical calculations: $\mu=\gamma=\varepsilon=0.01$; $\varphi=0.15$. The initial conditions correspond to the analytical solution (38): $u_1(0) = \varepsilon\bar{u}_1(W_*); w(0) = W_*; \dot{u}_1(0) = \dot{w}(0) = 0.$
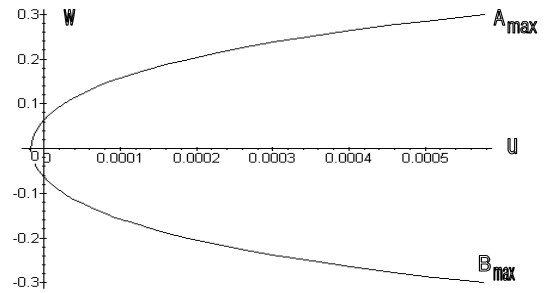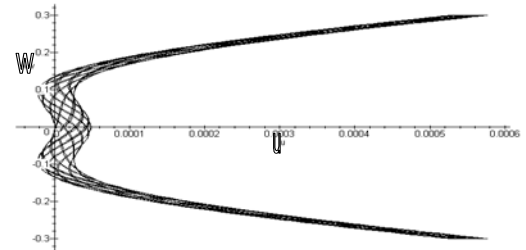

Fig. 4,a

Fig. 4,b

Fig.4. Periodic motion with large amplitude: 4,a) analytical solution; 4,b) numerical simulation with initial conditions, which correspond to the analytical results.

Fig. 4a shows the NNM, which is obtained according to formulae (38, 40). Fig. 4b shows the corresponding numerical results. As we can see from the Fig.3, the snap-through truss has significant amplitudes of oscillations and the main elastic system has small amplitudes. If such motions are stable, it guarantees the vibration absorption.

5.4. *The periodic motions stability*

5.4.1. *The stability of the periodic motions with small amplitudes*

For the stability investigation of the non-localized normal mode the new variables are introduced: $(\xi, \eta) = \left(u_1/g ; -u_1/g + w_1 \right)$. The ξ axis is directed along the rectilinear approximation of the NNM trajectory and η axis has the orthogonal direction. The system (35) with respect to the above-mentioned variables has the following form:

$$\ddot{\xi} + \xi + \varepsilon f_1(\xi, \eta) = 0;$$

$$\ddot{\eta} + 2s^2 p^2 \eta + \varepsilon f_2(\xi, \eta) = 0; \qquad (41)$$

where $f_1(\xi, \eta) = \bar{\gamma}c^2 \frac{p^2 s^2 - 1}{2p^2 s^2 - 1}\xi - \frac{\bar{\gamma}cs}{g}\eta;$

$$f_2(\xi,\eta) = -\bar{\gamma}c^2 \frac{p^2 s^2 - 1}{2p^2 s^2 - 1}\xi + \frac{\bar{\gamma}cs}{g}\eta + p^2(A\xi^2 + B\xi\eta + 3sc^2\eta^2);$$

$$A = 3sc^2 + \frac{1}{2}\tilde{\beta}g^2 + 2\tilde{\alpha}g; B = 6sc^2 + 2\tilde{\alpha}g.$$

The solution in time along the NNM trajectory is approximated as $\xi = \xi_{max}Cos(t) + O(\varepsilon)$, where ξ_{max} is the oscillations amplitude. Small perturbations $\Delta\eta(t)$ are added to periodic motion $\eta(t)$. Orthogonal variation $\Delta\eta(t)$ defines the NNM orbital stability. Therefore, the problem is reduced to the analysis of a single linear variational differential equation, which can be obtained from the equation (41). The variational equation to within the terms of order $O(\varepsilon)$ can be written in the form of the Mathieu's equation:

$$\Delta\ddot{\eta} + \left(2s^2 p^2 + \varepsilon\frac{\bar{\gamma}cs}{g} + \varepsilon p^2 B\xi_{max}Cos(t)\right)\Delta\eta = 0.$$

$$(42)$$

The known boundaries of the main region of instability in the system parameter space are the following: $2\sqrt{2}ps^2 - 8p^2 s^3 = \mp\varepsilon\xi_{max}$. The shaded region of instability is shown in Fig.5, where s=sinφ is plotted on x-axis. In region of the non-localized NNM instability the transfer to the through truss motion is taken place.
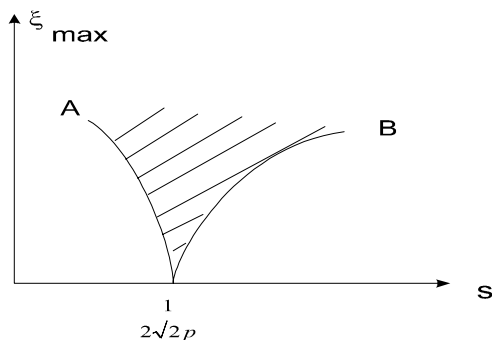

Fig. 5. Region of instability for the non-localized NNM

Additional analysis [36] shows that the variational equation for the localized NNM with small amplitude for the harmonic approximation of the solution can be wrote of the form:

$$\Delta\ddot{\eta} + \left[2p^2 s^2 + \varepsilon\bar{\gamma}c^2 + \varepsilon^2\bar{\gamma}\beta W_*^{(max)}Cos(ps\sqrt{2}t)\right]\Delta\eta = 0.$$

$$(43)$$

The following conclusions can be made. There are not resonances of order $O(\varepsilon^2)$, and resonances of order $O(\varepsilon^4)$ terms arise. So, the localized NNM is always stable to within order $O(\varepsilon^4)$.

5.4.2. *The stability of the periodic motions with large amplitudes*

One considers the stability of the absorption mode (38) with large amplitudes of the absorber. The variable u_1 is of order $O(\varepsilon)$ with respect to the variable w, so, we select from the system (5.2) the single equation to analyze variations which are orthogonal to the NNM. One introduces the small perturbations $\eta(t)$ of periodic motions $\bar{u}_1$:
$u_1 = \bar{u}_1 + \eta$. As a result, the following equation can be obtained from the second equation (5.2):

$$\ddot{\eta} + (1 + \varepsilon\bar{\gamma} - \frac{\varepsilon\bar{\gamma}}{\rho^3}w_0^2)\eta = 0 \qquad (44)$$

Here we use the following representation: w=w₀+O(ε). One has

$$\ddot{w}_0 - p^2\alpha^2 w_0 + \frac{p^2\beta^2}{2}w_0^3 = 0, \qquad \text{and}$$

$$w_0 = \sqrt{2}\frac{\alpha}{\beta}\sqrt{1 + \sqrt{1 + 4H}}\, cn(\, p\alpha\tau\sqrt[4]{1 + 4H}\,;k\,), \quad (45)$$

where

$$4H = \frac{\beta^4 W_*^4}{4\alpha^4} - \frac{\beta^2}{\alpha^2}W_*^2; \qquad (46)$$

$$2k^2 = \left(1 + \frac{\beta^4 W_*^4}{4\alpha^4} - \frac{\beta^2}{\alpha^2}W_*^2\right)^{-1/2} + 1. \quad (47)$$

Note that the last formula defines the amplitude value W_*. The following notations are taken in relations (45-46): k is a modulus of elliptic integral; cn(.) is an elliptic function; H is a total energy of the oscillator which is determined by the equation (46). The next relations are derived from the formula (47):

$$k^2 = k_0^2 - \varepsilon k_1 + O(\varepsilon^2); \qquad (48)$$

$$k_1 = \frac{\overline{\gamma}\left(1.5 - c^{-1}\right)\left[W_*^4 - 2c^2(1-c)W_*^2\right]c^2(1-c)}{\left[4c^4(1-c)^2 + W_*^4 - 4c^2(1-c)W_*^2\right]^{3/2}};$$

$$k_0^2 = 0.5 + \frac{c^2(1-c)}{\sqrt{4c^4(1-c)^2 + W_*^4 - 4c^2(1-c)W_*^2}}.$$

Substituting the Fourier-series expansion of $cn^2(t,k_0)$ into the equation (44), we can obtain the next Hill equation:

$$\ddot{\eta} + \left[\Omega_0^2 - \varepsilon h \sum_{s=1}^{\infty} \frac{sq_0^s}{(1 - q_0^{2s})} Cos(\Omega_* st)\right]\eta = 0; \quad (49)$$

$$h = \overline{\mu}\frac{4\pi^2}{K^2(k_0)}; \Omega_* = \frac{\pi}{K(k_0)}; q_0 = \exp\left[-\frac{\pi K'(k_0)}{K(k_0)}\right];$$

$$\Omega_0^2 = \frac{c(2k_0^2 - 1)}{2(1-c)p^2} - \frac{\varepsilon k_1 c}{(1-c)p^2} +$$

$$\frac{\varepsilon\overline{\mu}(2k_0^2 - 1)(1 + 2c)}{4(1-c)} - \varepsilon 2\overline{\mu}\left[\frac{E(k_0)}{K(k_0)} - 1 + k_0^2\right],$$

where $K(k_0)$, $E(k_0)$ are the complete elliptic integrals of the first and the second kinds.

The equation (49) is analyzed by the multiple scales method [55]. Solutions of this equation is chosen in the form:

$$\eta = \eta_0(T_0, T_1) + \varepsilon\eta_1(T_0, T_1) + ...; T_0 = t; T_1 = \varepsilon t;$$

$$\Omega_0^2 = \omega_0^2 + \varepsilon\omega_1^2 + ...$$

Then it can present:

$$\eta_0 = A(T_1)exp(i\omega_0 T_0) + \overline{A}(T_1)exp(-i\omega_0 T_0).$$

After some transformations and by using change of

variables, $A = \frac{a}{2}\exp(i\beta)$; $\omega_0 T_0 = \sigma_s T_1 - 2\beta$, we can write the next system of modulation equations:

$$a'\omega_0 = a\chi Sin(\sigma_s T_1 - 2\beta);$$

$$\beta'\omega_0 = \frac{\omega_1^2}{2} - \chi Cos(\sigma_s T_1 - 2\beta), \quad (50)$$

where $\chi = \frac{hsq_0^s}{4(1 - q_0^{2s})}$. Additional analysis [36] shows

that the following relation represents boundaries of the stable/unstable oscillations regions on the plane of the system parameters:

$$\Omega_* s = 2\omega_0 + \varepsilon\left(\frac{\omega_1^2}{\omega_0} \pm \frac{2\chi}{\omega_0}\right) + O(\varepsilon^2). \quad (51)$$

Besides,

$$W_* = \frac{2c^{3/2}}{p\pi s}\left[K\left(\frac{1}{\sqrt{2}}\right) + \varepsilon_*\frac{c^2c_1^2}{W_*^2}E\left(\frac{1}{\sqrt{2}}\right)\right] + O(\varepsilon_*^2) + O(\varepsilon).$$

(52)

Since the shallow snap-through truss is considered, the following notations were introduced:
$1 - c = \varepsilon_* c_1; \varepsilon_* \prec\prec 1.$

Fig. 6 shows curves $(OA_1),(OA_2),(OA_3)$ on plane (c, W_*), which are meet the equation (51). The boundaries of the stable/unstable regions are shown qualitatively on Fig. 6 in the form of curves $(B_1C_1D_1)$, $(B_2C_2D_2)$, $(B_3C_3D_3)$. Values $\widetilde{c}_s; s = 1,2,...$, which are shown on Fig. 6, are derived from the equations (51, 52). These values are determined according to the formula:

$$\widetilde{c}_s = \left[1 + \frac{4K^2\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2} - 1\right)}{p^2\pi^2 s^2}\right]^{-1}. \quad (53)$$

Magnitudes $\widetilde{c}_s$ have the following values: $\widetilde{c}_1 = 0.63; \widetilde{c}_2 = 0.88; \widetilde{c}_3 = 0.94$ for the system parameters chosen earlier in this section. In this case points A_i; (i=1,2,3), shown on Fig. 6, have the following coordinates: $A_1(1;1.18); A_2(1;0.59); A_3(1;0.39)$.
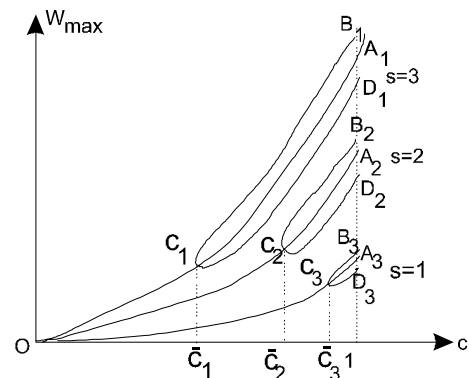

Fig. 6. Stable/unstable regions on plane (c, W_*) for the NNM

with large amplitudes. The boundaries of the regions are shown qualitatively as curves $(B_1C_1D_1)$, $(B_2C_2D_2)$, $(B_3C_3D_3)$. The curves $(OA_1),(OA_2),(OA_3)$ meet equation (20).

The following conclusions can be made from the Fig. 6. If the angle φ is small, the unstable oscillations regions have an order $O(\varepsilon)$. If a value φ is increased, the width and number of the unstable regions are increased too. It can choose such values of φ, that the periodic motions under

consideration are always stable. So, it can conclude that the localized NNM is the preferable regime to absorb oscillations. In this regime the snap-through truss has significant oscillation amplitudes, and the main linear subsystem has small amplitudes. Besides, the localized NNM mode is stable over a wide range of the system parameters change.

Forced oscillations of a system, containing the snap-through truss, close to its equilibrium position, were investigated in [37]. External time-periodic force $\varepsilon f cos(\omega t)$ acts on big mass M of the linear subsystem, and a snap-through truss is an absorber of the forced oscillations of the linear subsystem. The instability regions of non-localized forced oscillations are obtained. In these regions the non-localized oscillation amplitudes are increasing, and the absorber can fall into the snap-through motions, which are effective for the absorption of oscillations.

The snap-through forced motions, which are the most appropriate for absorption of the linear vibrations, was analyzed in [38]. The absorption vibration mode is constructed by using the the NNMs and Rauscher methods, and the asymptotical analysis. This mode was derived analytically. Numerical simulation shows a good accuracy of the analytical construction. It was stated that significant vibration amplitudes of the main linear subsystem in the system with the snap – through absorber are not observed in the frequency range of the resonance of the main linear subsystem without the absorber. The stability of the absorption mode was studied by using the multiple scales method. The stability analysis shows that the vibration absorption mode is stable for almost all values of the system parameters, excepting the very narrow resonance region. Therefore, the snap – through absorber is effective.

6. NONLINEAR OSCILLATIONS OF THE CYLINDRICAL SHELLS WITH INITIAL IMPERFECTIONS, IN A SUPERSONIC FLOW

A lot of studies were devoted to large amplitude vibrations of circular cylindrical shells. In some of them the nonlinear oscillations of the non-ideal cylindrical shells, that is cylindrical shells with initial imperfections, were considered. It can select some review-type publications on the subject, in particular, papers by Budiansky and Hutchinson [56], Kubenko and Kovalchuk [60]. Comparison of theoretical and experimental results on this subject is presented in [57] There are too a great number of papers which are available to the dynamics of cilindrical shells in the supersonic flow [58-61 et al.]. It was shown in paper [62] that initial deflections, which can be appeared in process of the shell preparation, substantially affect to static and dynamic characteristics of the shell in the supersoniv flow. In the book [63] the dependence of the shell frequencies on a form of the shell displacements and on the flatter boundary in the presence of the initial imperfections is obtained. It was shown that initial imperfections can

excite the vibration modes with other wave numbers than in a case of the ideal shell. The NNMs conception was used in [40] to investigate the non-ideal cylindrical shells dynamics, including the non-ideal shells dynamics in supersonic flow.

6.1. Free oscillations of shells with initial imperfections

Free nonlinear oscillations of the circular cylindrical shells with imperfections are described by the well-known Donnell-Mushtari equations:

$$\frac{D}{h}\nabla^4 w_1 = L(w_1 + w_0, \Phi) + \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} - \rho\frac{\partial^2 w_1}{\partial t^2} \quad (54)$$

$$\frac{1}{E}\nabla^4 \Phi = -\frac{1}{2}L(w_1 + 2w_0, w_1) - \frac{1}{R}\frac{\partial^2 w_1}{\partial x^2}, \quad (55)$$

where the differential operator

$$L(A,B) = \frac{\partial^2 A}{\partial x^2}\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 A}{\partial y^2}\frac{\partial^2 B}{\partial x^2} - 2\frac{\partial^2 A}{\partial x \partial y}\frac{\partial^2 B}{\partial x \partial y};$$

$$\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

Here E is the Young modulus; μ is the Poisson ratio; ρ is the shell density; D is the cylindrical rigidity of the shell, R, h are the radius and thickness of the shell; x,y are longitudinal and circumference coordinates; w_1 is the normal dynamical deflection; Φ is the in-plane Airy stress function. Normal initial imperfections of the shell associated with zero initial tension are denoted by radial displacements $w_0(x,y)$ in the equations (54, 55). So, the complete normal deflection of the shell is $w = w_1 + w_0$. It is assumed that stresses in the axial and circumference directions, $\overline{N}_x = 0, \overline{N}_y = 0$.

Then the shell oscillations represented in the following form:

$$w_1 = f_1(t)\sin rx \sin sy + f_2(t)\sin rx \cos sy + f_3(t)\sin^2 rx \quad (56)$$

Here $s = n/R, r = m\pi/L$; n is a number of waves in the circumference direction; m is a number of half-wave in the longitudinal direction; unknown functions $f_1(t), f_2(t)$ describe two asymmetric modes; $f_3(t)$ is a general coordinate, which characterizes the axisymmetric mode. It is assumed that a length of the middle surface transverse section is constant during oscillations:

$$\int_0^{2\pi R}\left(-\frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)dy = 0 \quad (57)$$

In this case, using the relation (57), the next relation can be derived: $f_3 = \dfrac{n^2}{4R}(f_1^2 + f_2^2)$.

It is assumed that imperfections are of the following form, which corresponds to the first principal asymmetric mode:

$$w_0 = f_{10}\, sin\, rx\, sin\, sy + f_{20}\, sin\, rx\, cos\, sy \quad (58)$$

Substituting relations (56, 58) into (55), it is not difficult to find out the in-plane Airy stress function Φ which is not presented here. Using the Bubnov-Galerkin approach to the equation (6.1), the differential equations of oscillations can be derived as:

$$\ddot{f}_1 + \omega_1^2 f_1 + \gamma f_2 + 2\chi f_1(\dot{f}_1^2 + f_1\ddot{f}_1 + \dot{f}_2^2 + f_2\ddot{f}_2) +$$
$$\gamma_1 f_1(f_1^2 + f_2^2) + \alpha_1 f_1 f_2 + \alpha_2 f_1^2 + \alpha_3 f_2^2 = \omega_0^2 f_{10} \quad (59)$$
$$\ddot{f}_2 + \omega_2^2 f_2 + \gamma f_1 + 2\chi f_2(\dot{f}_1^2 + f_1\ddot{f}_1 + \dot{f}_2^2 + f_2\ddot{f}_2) +$$
$$\gamma_1 f_2(f_1^2 + f_2^2) + \beta_3 f_1 f_2 + \beta_1 f_1^2 + \beta_2 f_2^2 = \omega_0^2 f_{20}$$

where $2\chi = \dfrac{3}{2}\left(\dfrac{n^2}{2R}\right)^2$; $\gamma = -\dfrac{Er^4}{8\rho}f_{10}f_{20}$;

$\omega_1^2 = \omega_0^2 - \dfrac{Er^4}{16\rho}(f_{10}^2 - f_{20}^2)$;

$\omega_2^2 = \omega_0^2 + \dfrac{Er^4}{16\rho}(f_{10}^2 - f_{20}^2)$;

$\omega_0^2 = \dfrac{1}{\rho}\left[\dfrac{D}{h}(r^2 + s^2)^2 + \dfrac{Er^4}{R^2(r^2 + s^2)^2}\right]$;

$\gamma_1 = \dfrac{1}{\rho}\left[\dfrac{E}{16}r^4 + \dfrac{Dn^4 r^4}{hR^2} - \dfrac{Er^4 s^4}{(r^2 + s^2)^2}\right]$;

$\alpha_1 = \dfrac{Er^4 s^4}{2\rho(r^2 + s^2)^2}f_{20}$; $\alpha_2 = \dfrac{3Er^4 s^4}{4\rho(r^2 + s^2)^2}f_{10}$;

$\alpha_3 = \dfrac{Er^4 s^4}{4\rho(r^2 + s^2)^2}f_{10}$; $\beta_1 = \dfrac{Er^4 s^4}{4\rho(r^2 + s^2)^2}f_{20}$;

$\beta_2 = \dfrac{3Er^4 s^4}{4\rho(r^2 + s^2)^2}f_{20}$; $\beta_3 = \dfrac{Er^4 s^4}{2\rho(r^2 + s^2)^2}f_{10}$.

We can note that if the initial imperfections are equal to zero, all modal lines in the configurational space of the system (59) are rectilinear, and a number of these NNMs are equal to infinity. So, it is advisable in the system (59) for a case of the small initial imperfections to use the straight line to approximate the NNMs in the system:

$$f_2 = kf_1 \quad (60)$$

Substituting the relation (60) to the system (59), one has a condition of compatibility, which is a single nonlinear algebraic equation with respect to k. The amplitude value X of the general coordinate x is substituted instead of the coordinate into this algebraic equation. As a result, the following equation is derived:

$$X\,[\,k(\Omega_1^2 - \Omega_2^2) + \gamma(k^2 - 1)] +$$
$$X^2\left[\dfrac{\beta_3}{2}k^3 + (2\beta_1 - \beta_2)k^2 + (\alpha_2 - \beta_3)k - \beta_1\right] = \quad (61)$$
$$\omega_0^2(kf_{10} - f_{20}),$$

(6.5)

where $\Omega_{1,2}^2 = \omega_0^2 - \omega_{1,2}^2$.

The equation (61) is an algebraic cubic equation with respect to k. One considers now some limiting cases for different values of imperfections f_{10}, f_{20} and the amplitude X.

I. The linear limit, $X \to 0$. One has from the equation (61) only a single limiting NNM in a form (60), with $k = f_{20}\,/\,f_{10}$.

II. The symmetrical case, $f_{10} = f_{20}$. In this case one has that one of the roots of the equation is always equal to $k = 1$.

III. The limiting asymmetric case, $f_{20} = 0$. In this case one of the equation (61) roots is always equal to $k = 0$, and other roots can be calculated from the relation $k^2 = \dfrac{\omega_0^2 f_{10} - 2X\Omega_1^2}{X^2\alpha_3} - 1$.

IV. In a case of the small initial imperfections, that is, in the limit $f_{10}, f_{20} \to 0$, one has that the limiting system permits NNMs in the form (60) for any k. If f_{io} are small, but are not equal to zero, one has always from the equation (61) the root which is close to $k = f_{20}\,/\,f_{10}$.

Then the nonlinear normal mode trajectory can be presented in the form of the power series by the small parameter. The analytical and numerical analysis are shown that in the wide range of the system parameters only a single stable nonlinear vibration mode exists. The system trajectory is close to (60), where $k = f_{20}\,/\,f_{10}$. The obtained NNMs are verified by numerical integration of the system (596). Two examples of the stable NNMs are shown in Fig.7, where

$R = 0.2, L = 0.4, h = 0.5*10^{-3}, f_{10} = 0.1*10^{-3}$,
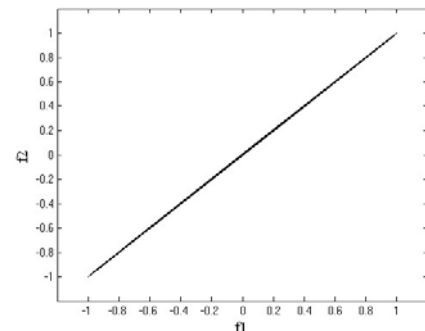
$E = 2*10^{11}, \rho = 8905, \mu = 0.35, n = 5$

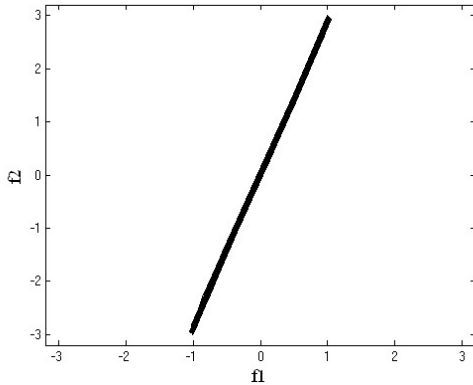Fig.7a. The modal line for $k = 1$

Fig.7b. The modal line for $k = 3$.

Fig.7. The stable NNM trajectories of the shell model (59)

6.2. The NNMs of circular shells without initial imperfections in a supersonic flow

One considers the nonlinear normal modes approach to analyze dynamics of cylindrical shells in a supersonic flow and a construction of the corresponding periodic solutions with large amplitudes. The Donnell-Mushtari equations (54, 55) are used here to study the circular shells dynamics without initial imperfections $(w_0(x,y)=0)$ in a supersonic flow. The radial aerodynamic pressure P describes the external supersonic flow, which acts on the shell surface. A first-order linear piston theory [59, 60] is used to approximate the pressure:

$$P = \chi P_\infty [\frac{1}{a_\infty} \frac{\partial w}{\partial t} + M_\infty \frac{\partial w}{\partial x}] \qquad (62)$$

where χ is the polytrophic exponent; a_∞ is the free stream of sound; P_∞ is the free stream static pressure; M_∞ is the Mach number.

The shell deflection is presented as

$$w = f_1 \sin rx \sin sy + f_2 \sin rx \cos sy +$$
$$f_3 \sin 2rx \sin sy + f_4 \sin 2rx \cos sy + f^*(x,t) \qquad (63)$$

Using the inextensibility condition of a middle surface of the form (57), the following expression for f^* is derived:

$$f^*(x,t) = \frac{n^2}{4R} [(f_1 \sin rx + f_3 \sin 2rx)^2 \qquad (64)$$
$$+ (f_2 \sin rx + f_4 \sin 2rx)^2]$$

Then, substituting the presentation (63) into the equation (55), it is possible to obtain the in-plane Airy stress function Φ which is not presented here.

Using the Bubnov-Galerkin method for the equation (54), the initial continuous system is reduced to the discrete one. The discrete dynamical system is obtained in the form:

$$\ddot{f}_1 + \omega_1^2 f_1 + \alpha \dot{f}_1 - \beta f_3 = Z_1(f_i, \dot{f}_i, \ddot{f}_i)$$
$$\ddot{f}_2 + \omega_1^2 f_2 + \alpha \dot{f}_2 - \beta f_4 = Z_2(f_i, \dot{f}_i, \ddot{f}_i)$$
$$\ddot{f}_3 + \omega_2^2 f_3 + \alpha \dot{f}_3 + \beta f_1 = Z_3(f_i, \dot{f}_i, \ddot{f}_i)$$
$$\ddot{f}_4 + \omega_2^2 f_4 + \alpha \dot{f}_4 + \beta f_2 = Z_4(f_i, \dot{f}_i, \ddot{f}_i)$$

(65)

where ω_1, ω_2 are frequencies of shell linear modes; Z_1, Z_2, Z_3, Z_4 are nonlinear terms. The parameters α and β meet the following relations:

$$\alpha = \frac{\chi P_\infty}{\rho h a_\infty}; \ \beta = \frac{8}{3} \frac{\chi P_\infty M_\infty}{Lh\rho}; \qquad (66)$$

The region of the flatter is considered. Let us consider the linear case, $Z_i = 0$, of the system (65). In the linear approximation the linearized normal mode of vibration of the form $f_1 = f_2, f_3 = f_4$ exists (Fig.8,a). Dependence between the coordinates f_1 and f_3 is more complicated. Corresponding trajectories in the place of the coordinates are the closed oval-type ones (Fig 8,b). As follows from the results of numerical simulations, the nonlinearity in the system (65) slightly deforms the NNM trajectories in the linear system. So, the periodic limit circle of the form of the NNM in region of the flatter is obtained.

But on the large calculation interval the obtained limit circle of the nonlinear system (65) becomes unstable. Namely, on the large interval of calculation on can observe a transfer to regime of beating, where amplitudes of beatings are about twice more than ones of the unstable limit circle. Analogical regime of vibrations was obtained experimentally by Stearman [60].
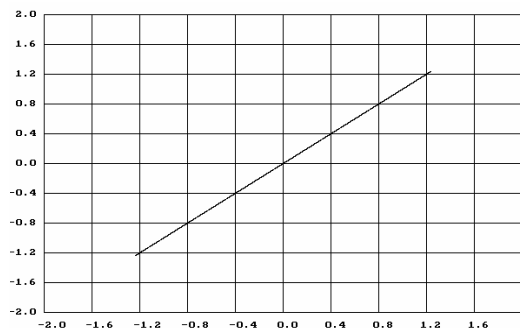

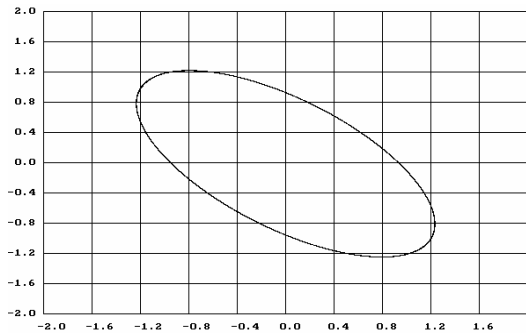
Fig.8,a. The NNM trajectory in the plane f_1, f_2

Fig.8,b. The NNM trajectory in the plane f_1, f_3

Fig.8. Trajectories of the NNMs in region of a flatter.

Additional investigation of the periodic regimes in the system under consideration was made by the harmonic balance method. A system of obtained algebraic equations was used by the Gauss-Newton method. The frequency response for the amplitude a_1 of the variable f_1 is shown in the Fig. 9. We can see here two obtained very close brahches. It demonstrates that the beating must be realized in this case.
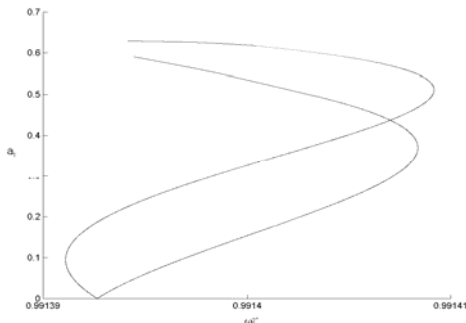
Fig.9. The frequency response for the amplitude a_1 **of the**

variable f_1**, obtained by the harmonic balance method.**

Other computer simulation fully corresponds to the preceding results.

6.3. The nonlinear dynamics of circular shell with initial imperfections in a supersonic flow

One considers now the combined influence of the initial imperfections and the supersonic flow to the shell nonlinear dynamics.

Here the following four modes approximation is chosen:

$$w = (f_1 - f_{10}) \sin rx \sin sy + (f_2 - f_{20}) \sin rx \cos sy +$$
$$f_3 \sin 2rx \sin sy + f_4 \sin 2rx \cos sy +$$
$$\frac{n^2}{4R} [(f_1 \sin rx + f_3 \sin 2rx)^2 + (f_2 \sin rx + f_4 \sin 2rx)^2]$$

(67)

Initial imperfections are chosen of the form, which corresponds to the first principal asymmetric mode:

$$w_0 = f_{10} \sin rx \sin sy + f_{20} \sin rx \cos sy \quad (68)$$

Then after determination of the Airy stress function, and using the Bubnov-Galerkin procedure, one has the following system of the nonlinear ODEs, which contains the unknown functions $f_i(t), i = 1,2,3,4$.

$$\ddot{f}_1 + \omega_{11}^2 f_1 + \alpha \dot{f}_1 - \beta f_3 - \omega_1 f_{10} + \alpha_1 f_1^2 +$$
$$\alpha_2 f_2^2 + \alpha_3 f_3^2 + \alpha_4 f_1 f_2 + \alpha_5 f_3 f_4 - \alpha_6 f_2 = Z_1$$
$$\ddot{f}_2 + \omega_{12}^2 f_2 + \alpha \dot{f}_2 - \beta f_4 - \omega_1 f_{20} + \beta_1 f_1^2 +$$
$$\beta_2 f_2^2 + \beta_3 f_4^2 + \beta_4 f_1 f_2 + \beta_5 f_3 f_4 - \beta_6 f_1 = Z_2 \quad (68)$$
$$\ddot{f}_3 + \omega_{21}^2 f_3 + \alpha \dot{f}_3 + \beta f_1 + \gamma_1 f_1 f_3 + \gamma_2 f_1 f_4 +$$
$$\gamma_3 f_2 f_4 - \gamma_4 f_4 = Z_3$$
$$\ddot{f}_4 + \omega_{22}^2 f_4 + \alpha \dot{f}_4 + \beta f_2 + \sigma_1 f_1 f_3 + \sigma_2 f_2 f_3 +$$
$$\sigma_3 f_2 f_4 - \sigma_4 f_3 = Z_4$$

Here $\alpha_1 = \frac{3}{4} \delta f_{10}; \quad \alpha_2 = \frac{1}{4} \delta f_{10}; \quad \alpha_3 = \frac{5}{4} \delta f_{10};$

$$\alpha_4 = \frac{1}{2} \delta f_{20}; \qquad \alpha_5 = \frac{5}{4} \delta f_{20}; \alpha_6 = \delta_1 f_{10} f_{20};$$

$$\beta_1 = \frac{1}{4} \delta f_{20}; \qquad \beta_2 = \frac{3}{4} \delta f_{20}; \qquad \beta_3 = \frac{5}{4} \delta f_{20};$$

$$\beta_4 = \frac{1}{2} \delta f_{10}; \qquad \beta_5 = \frac{5}{4} \delta f_{10}; \qquad \beta_6 = \delta_1 f_{10} f_{20};$$

$$\gamma_1 = \frac{5}{2} \delta f_{10}; \qquad \gamma_2 = \frac{5}{4} \delta f_{20}; \qquad \gamma_3 = \frac{5}{4} \delta f_{10};$$

$$\gamma_4 = 4\delta_1 f_{10} f_{20}; \qquad \sigma_1 = \frac{5}{4} \delta f_{20}; \qquad \sigma_2 = \frac{5}{4} \delta f_{10};$$

$$\sigma_3 = \frac{5}{2} \delta f_{20}; \ \sigma_4 = 4\delta_1 f_{10} f_{20}; \ \delta = \frac{Er^4 s^4}{\rho(r^2 + s^2)^2};$$

$$\delta_1 = \frac{Er^4}{8\rho}; \ \alpha = \frac{\chi P_\infty}{\rho h a_\infty}; \ \beta = \frac{8}{3} \frac{\chi P_\infty M_\infty}{Lh\rho};.$$

Frequencies of the linearized system are the following:

$$\omega_{11}^2 = \omega_1^2 - \frac{Er^4}{16\rho}(f_{10}^2 - f_{20}^2);$$

$$\omega_{21}^2 = \omega_1^2 + \frac{Er^4}{16\rho}(f_{10}^2 - f_{20}^2);$$

$$\omega_{23}^2 = \omega_2^2 - \frac{Er^4}{4\rho}(f_{10}^2 - f_{20}^2);$$

$$\omega_{24}^2 = \omega_2^2 + \frac{Er^4}{4\rho}(f_{10}^2 - f_{20}^2);$$

$$\omega_1^2 = \frac{1}{\rho}\left[\frac{D}{h}(r^2 + s^2)^2 + \frac{Er^4}{R^2(r^2 + s^2)^2}\right];$$

$$\omega_2^2 = \frac{1}{\rho}\left[\frac{D}{h}(4r^2 + s^2)^2 + \frac{4Er^4}{R^2(4r^2 + s^2)^2}\right].$$

The functions Z_i in equations (68), which are not

presented here in view of their awkwardness, are nonlinear with respect to the general deflections f_i $(i=1,2,3,4)$ and their derivatives. These functions do not contain initial imperfections.

For the numerical calculations some physical parameters were fixed, namely, $\rho=8905.37 кг/м^3$, $E=2*10^{11}Па$, $\mu=0.35$, $\chi=1.4$, $M_\infty=3$, $a_\infty=213 м/с$. In the Fig. 10 it is shown vibration regimes of the system (68) in time and corresponding trajectories in configuration space for two different ratio of the initial imperfections amplitudes, f_{10} and f_{20} In the Fig. 10,a the first three diagrams are relative to a case $k=0.5$, and the next three diagrams are relative to a case $k=5$. We can see regimes (in time) of beating, but trajectories are the same as for the nonlinear normal modes. Numerous numerical calculations confirm that in the system NNMs with near rectilinear trajectories in configuration space exist, as in the case when the flow influence is absent. These trajectories are close to the rectilinear trajectories

$$f_2 = kf_1 \quad \text{and} \quad f_4 = kf_3, \text{ where } k = f_{20}/f_{10}.$$ As for ideal shell the trajectory on the place f_1, f_3 for the linearized system represents some the closed oval-type trajectory (see Fig.8,b). In the complete nonlinear system this trajectory is unstable, and we can observe a transfer to regime of beating, where amplitudes of beatings are about twice more than ones of the unstable limit circle.

So, in the flatter region we can observe that the vibration modes, that is trajectories in the system configuration space, for the non-ideal shell are similar to ones in the non-ideal shell without the flow influence. Simultaneously, vibrations in time are analogous to ones in the ideal shell under the flow influence.
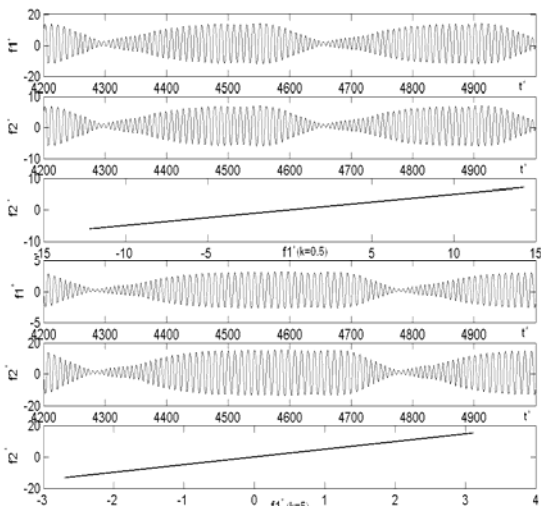

Fig.10,a. A case $k=0.5$ (three upper diagrams)
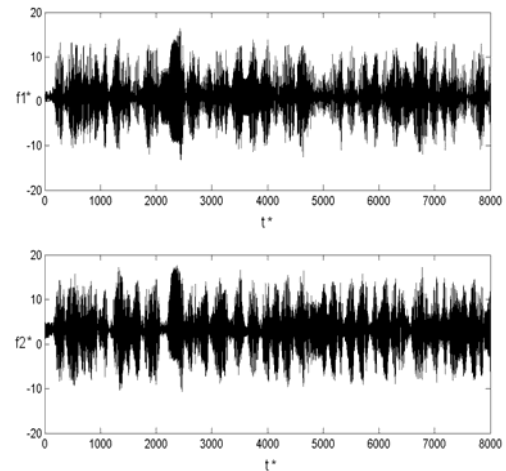
and a case $k=5$ (three lower diagrams)

Fig.10,b. Complicated dynamics for large initial imperfections amplitudes, $f_{10}=3$, $f_{20}=1$

Fig. 10. Vibrations in time of the system (68) and corresponding trajectories in configuration space (Fig.10,a) for different ratio of the initial amplitudes of imperfections, f_{10} and f_{20}

But these vibration regimes are not typical for large values of the initial imperfections. For large values of imperfections the vibration modes are not remained. It can see this situation in the Fig. 10,b, where the behavior of the system is presented for the next amplitudes of the initial imperfections: $f_{10}=3$, $f_{20}=1$. Numerous numerical experiments show that if the initial imperfection have an order more that the shell thickness, the complicated dynamics of the shell is typical. These dynamics is similar to chaotic one.

7. NNMs IN THE VEHIKLE DYNAMICS

There are a lot of publications on the dynamics of the linear vehicle models. However, in fact the system is nonlinear because it contains elastic components with nonlinear characteristics. Nonlinear effects in the suspension dynamics are important and must be considered if corresponding displacements are equal to 0.05-0.1 m, or larger. The nonlinear vehicle dynamics is modeled, as a rule, by simplest models, the "quarter-car" model (two-DOF system) for studying the heave motion [65, 66], or the "half-car" model (four-DOF system) for studying the heave and pitch motions [67, 68].

Nonlinear 7-DOF model of the vertical and axial vehicle dynamics is considered here for a case of independent-solid axle suspension to predict the vehicle body and wheel states. It is possible to study, by using this model, all principal vehicle motions, namely, the heave, pitch and roll motions [69-71]. It is supposed here that, in many cases, for example, after impact, the nonlinear normal modes are, may be, the most important regimes to describe the vehicle

nonlinear dynamics, because a transient is very short-term process due to a strong dissipation in the system under consideration. The NNMs approach can be successfully used as for smooth, and for non-smooth characteristics of the car suspension.

7. 1. The NNMs for the nonlinear 7-DOF model of the vehicle

In order to describe the vertical dynamics of a double-tracked road vehicle the 7-DOF model (Fig. 11) is used, which based on Refs. [68,69]. The car body is represented as a rigid body. The heave, roll and pitch motions are considered. Here z is the vertical displacement, α is the pitch angle, β is the roll angle, x_i are the vertical displacement of i-th suspended mass which are equivalent to the wheel, d_1, d_2 are front, rear track width and l_1, l_2 are front, rear wheel base. In this model tires are presented as elastic elements with linear characteristics. The suspension is characterized at first by nonlinear elastic characteristics of the front and rear springs, and linear damping characteristics. Some possible elastic characteristics are shown in the Fig. 12 (some information on such characteristics can be found, for example, in [71] and other publications). Elastic forces appearing in the springs ($f_1(x)$ and $f_2(x)$) can be correctly approximated by polynomials of the 7-th degree.
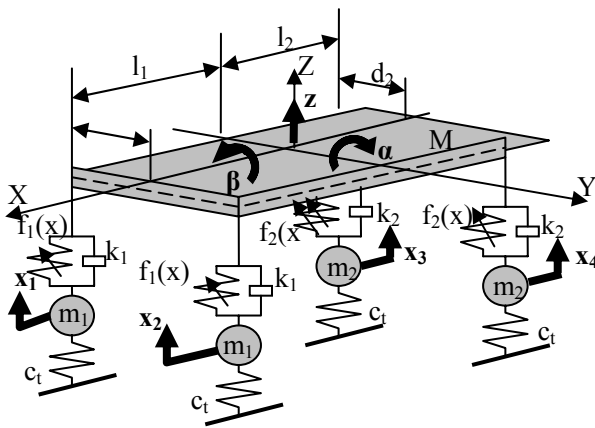

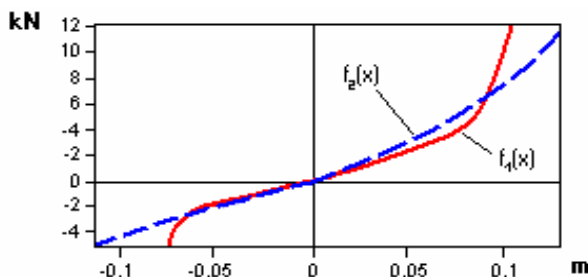**Fig. 11. Model of a double-tracked road vehicle under consideration**

Fig. 12. Typical nonlinear characteristics of the front $f_1(x)$ and rear $f_2(x)$ tracks.

One has seven generalized coordinates to describe vibrations of the model. Body center mass displacements are characterized by the vector $q = \{z \quad \alpha \quad \beta\}^T$, and displacements of the suspended masses which are equivalent

to the wheels, by the vector $x = \{x_1 \quad x_2 \quad x_3 \quad x_4\}^T$. Matrices of the body masses M_C, of the suspension elements M_S, of the tire stiffness C, and of the damping K are diagonal. Displacements of the body angles are connected with displacements of the mass center by the following matrix:

$$H = \begin{bmatrix} 1 & -l_1 & d_1 \\ 1 & -l_1 & -d_1 \\ 1 & l_2 & d_2 \\ 1 & l_2 & -d_2 \end{bmatrix}. \tag{69}$$

One introduces a difference of displacements of the body and elements of suspension as $U = Hq - x$, and difference of velocities as $V = H\dot{q} - \dot{x}$. A vector of nonlinear characteristics can be written of the next form:

$$C_{NL} = \{f_1(U_1) \quad f_1(U_2) \quad f_2(U_3) \quad f_2(U_4)\}^T, \tag{70}$$

where U_i are components of the vector U.

One has finally the following ODE system (in the matrix form) which describes free nonlinear vibrations of the car:

$$\begin{cases} M_C\ddot{q} + H^T C_{NL} + H^T KV = 0, \\ M_S\ddot{x} - C_{NL} + Cx - KV = 0. \end{cases} \tag{71}$$

The NNMs conception by Show and Pierre, which was presented in Section 4, is used here. The equations (71) can be wrote and the system solution in the form of the power series (33) by new independent variables u and v are used.

This procedure permits to obtain seven NNMs of the system under consideration. Few surfaces, which characterize the first NNM, and corresponding trajectories of motion on these surfaces for some concrete coordinates are shown in the Fig. 13. Here the coordinate z, describing the vertical displacement, is chosen as the independent variable u, and the corresponding velocity $\dot{z}$ is chosen as the independent coordinate v.

If the NNM in the form (33) is obtained, these series are substituted to equations of motion, and functions $u = u(t)$ and $v = v(t)$ can be obtained too. Numerical calculations show a good exactness of the obtained analytical results. Note that the values of the car parameters, which are presented in the Table 1, were used in calculations.

Table 1. Values of the car parameters

Parameter	Value	Parameter	Value
M	2369 kg	K_2	900 kg/s
I_x	4108kg m2	c_t	258000 n/m
I_y	938 kg m2	L_1	1,459 m

M_1	77 kg	L_2	1,486 m
M_2	108,5 kg	D_1	0,868 m
K_1	700 kg/s	D_2	0,837 m
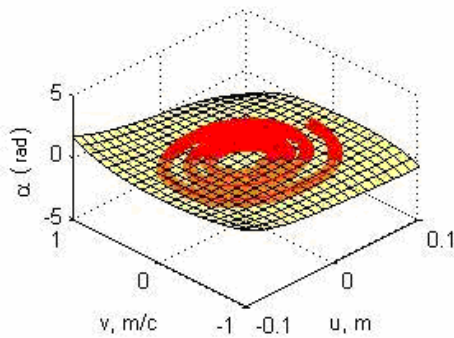
a)Change of the angle α depending on variables *u* and *v*
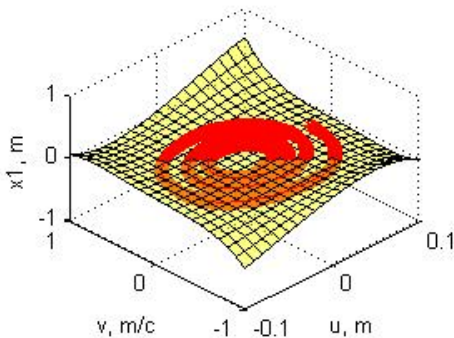
b) Change of the angle α velocity depending on variables *u* and *v*

Fig. 13. The first vibration mode of the nonlinear system (independent coordinates are u=z and v=$\dot{z}$; initial values: $z = 0,075$, and $\dot{z} = 0$)

To construct the NNMs skeletons the harmonic linearization method together with a continuation procedure can be used.

One can observe that in a case of large initial displacements a vibration frequency of the unsprung masses lowers quickly, and all nonlinear vibration modes tend to low-frequency vibration modes of the car body.

7.2. Nonlinear normal modes and transient (shock absorber, non-smooth characteristic)

To investigate the suspension dynamics taking into account a non-smooth characteristic of the shock absorber, the quarter-car model is considered (Fig. 14).

Equations of motion for the quarter-car model are the following:

$$\begin{cases} M\ddot{x}_1 + f(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = 0, \\ m\ddot{x}_2 + f(x_2 - x_1) + d(\dot{x}_2 - \dot{x}_1) + c_t x_2 = 0. \end{cases} \quad (72)$$

where $f(x)$ is a stiffness function, $d(\dot{x})$ is a piecewise damping function of the suspension, namely,

$$d(\dot{x}) = \begin{cases} d_1(\dot{x}), & \dot{x}_1 - \dot{x}_2 < 0, \\ d_2(\dot{x}), & \dot{x}_1 - \dot{x}_2 \geq 0. \end{cases} \quad (73)$$

Values of the car parameters, which were used in calculations, are the following:

$$M = 592.25 \text{ kg}, \quad m = 108.5 \text{ kg}, \quad c_t = 258000 n/m,$$
$$f(x) = 55000 x.$$

Characteristic of the damping functions for a case of piecewise linear damping is presented in Fig. 15. This is an approximation of more exact characteristic which will be considered later. The stiffness characteristic in the suspension is chosen here as linear.
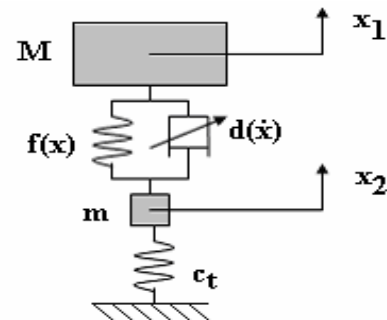

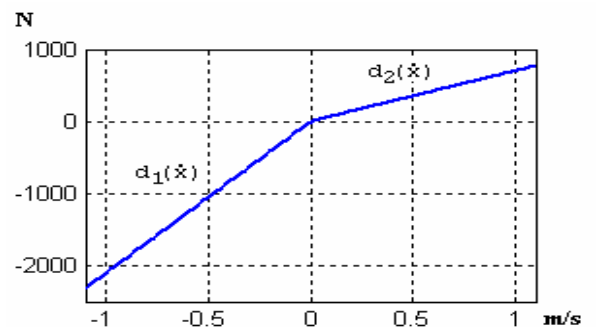Fig. 14. The quarter-car model

Fig. 15. The piecewise linear shock absorption characteristic in the suspension

The first NNM obtained by using the method described below are shown for some concrete values of the system parameters in Fig. 16. One can observe here as motions on places corresponding the NNMs, as well a transient from one place to another one after gap (or "switching") of the piecewise linear damping characteristic.
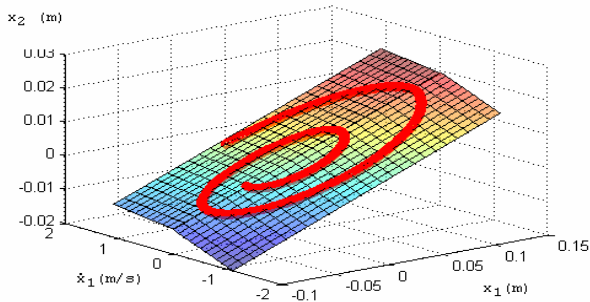
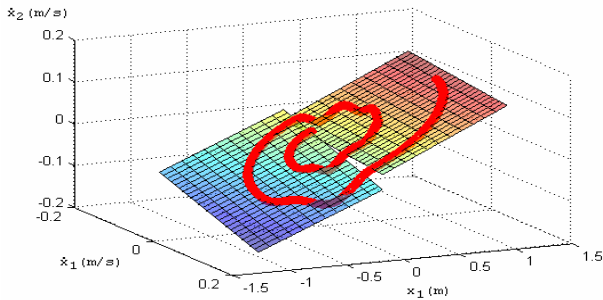**Fig 16, a. Change of displacement x_2 (depending on u and v)**

Fig. 16,b. Change of the displacement x_2 velocity (depending on u and v)

Fig. 16. The first NNM. Independent variables are x_1 and the corresponding velocity (the piecewise linear characteristic)

Fig. 19,a. Change of displacement x_2 (depending on u and v)

Fig. 19,b. Change of displacement x_2 velocity (depending on u and v)

Fig. 19. First NNM. Independent variables are x_1 and the corresponding velocity (the piecewise cubic characteristic)

More realistic nonlinear suspension characteristics for stiffness and absorption (for a piecewise cubic damping) are shown in Fig. 17. The first NNM, obtained in this case, and the transient from one surface to another one after gap (or "switching") in the piecewise cubic damping characteristic are shown in Fig. 18. Note that in all cases the dominant pair of phase variables is chosen as independent coordinates in the NNM analytical representation.
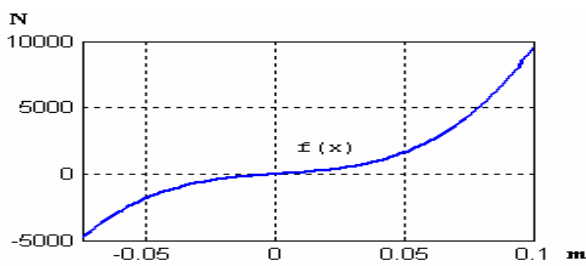

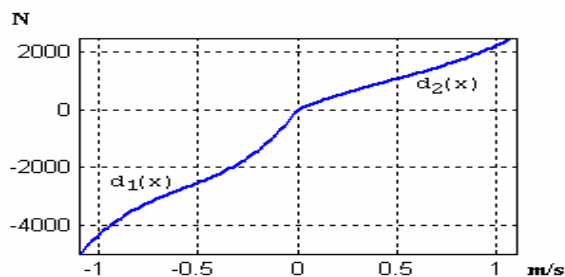**Fig. 17,a. The nonlinear cubic stiffness characteristic in the suspension**

Fig. 17,.b. The piecewise cubic shock absorption characteristics in the suspension

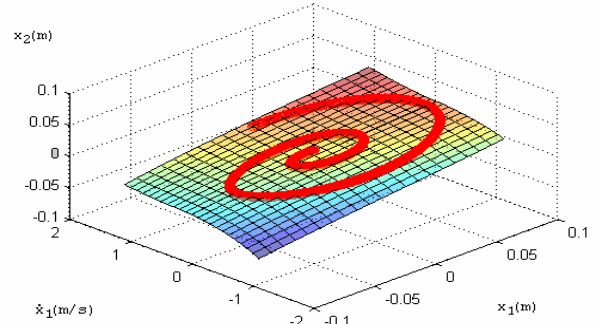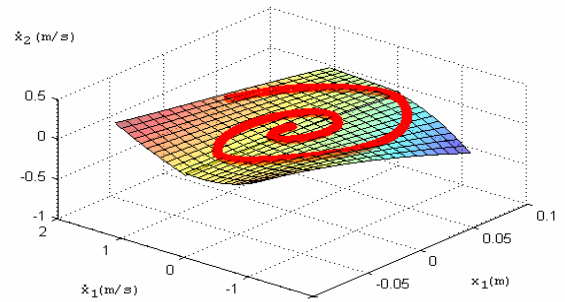**Fig. 17. The nonlinear suspension characteristics for stiffness and shock absorption**

3. CONCLUSION

Nonlinear normal modes (NNMs) are typical regimes which exist in different classes of conservative or near-conservative many-DOF systems. Two principal conceptions of NNMs are considered. The Kauderer-Rosenberg conception, when all positional coordinates are single-valued functions of some of them, is associated with trajectories in configurational space. The Shaw-Pierre conception is is based on the computation of invariant manifolds of motion. In this case the NNMs can be obtained as single-valued functions of two selected phase coordinates. The Shaw-Pierre NNMs are very effective is quasilinear dissipative systems. An efficiency of the NNMs theory is shown in some applied problems, in particular, in analysis of nonlinear dynamics of shells, in the vibro-absorption nonlinear problem and in problem of the vehicle suspension nonlinear dynamics.

REFERENCES

[1] A. M. Lyapunov, The general problem of the stability of motion, Princeton University Press, Princeton, 1947.

[2] H. Kauderer, Nichtlineare mechanik. Springer-Verlag, Berlin, 1958.

[3] Seifert H. Periodische Bewegungen mechaischen Systeme// Math. Z., Vol. 51. pp.197-216, 1948.

[4] R. Rosenberg, The normal modes of nonlinear n-degrees-of-freedom systems, Journal of Applied Mechanics, 30, pp. 7–14, 1942.

[5] R. Rosenberg, Nonlinear vibrations of systems with many degrees of freedom, in Advances of Applied Mechanics, Vol. 9, Academic Press, New York, pp. 156–243, 1966.

[6] R. Rosenberg and J. Kuo, Nonsimilar normal mode vibrations of nonlinear systems having two degrees of freedom, Journal of Applied Mechanics, 31, pp. 283–290, 1964.

[7] R. Rand, A higher order approximation for nonlinear normal modes in two degrees of freedom systems, International Journal of Non-Linear Mechanics, 6, pp. 545–547, 1971.

[8] L. Manevich and Yu. Mikhlin, Periodic solutions close to rectilinear normal vibration modes, Prikladnaja Matematica i Mekhanica (PMM USSR), 36, pp. 1051–1058, 1972.

[9] Yu. Mikhlin, Normal vibrations of a general class of conservative oscillators, Nonlinear Dynamics, 11, pp. 1–16, 1966.

[10] L. Manevich, Yu. Mikhlin and V. Pilipchuk, The method of normal oscillation for essentially nonlinear systems, Nauka, Moscow, 1989 (in Russian).

[11] E. Vedenova, L. Manevich and V. Pilipchuk, Normal oscillations of a string with concentrated masses on non-linearly elastic support, Prikladnaja Matematica i Mekhanica (PMM USSR), 49, pp. 203–211, 1985.

[12] A. Vakakis, Non-similar normal oscillations in a strongly non-linear discrete system, Journal of Sound and Vibration, 158, pp. 341–361, 1992.

[13] A. Vakakis and R. Rand, Normal modes and global dynamics of a two-degree-of freedom non-linear system. I. Low energies. II. High energies, International Journal of Non-Linear Mechanics, 27, pp. 861–888, 1992.

[14] Yu. Mikhlin, Matching of local expansions in the theory of nonlinear vibrations, Journal of Sound and Vibration, 182, pp. 577–588, 1995.

[15] S. Shaw and C. Pierre, Nonlinear normal modes and invariant manifolds, Journal of Sound and Vibration, 150, pp. 170–173, 1991.

[16] S. Shaw and C. Pierre, Normal modes for nonlinear vibratory systems, Journal of Sound and Vibration, 164, pp. 85–124, 1993.

[17] I. Malkin, Certain problems of the theory of nonlinear vibrations. Geotechteorizdat, Moscow, 1956, 492 pps, 1956.

[18] W. Kinney and R. Rosenberg, On the steady state vibrations of nonlinear systems with many degrees of freedom, Journal of Applied Mechanics, 33, pp. 406–412, 1966.

[19] Yu. Mikhlin, Resonance modes of near-conservative nonlinear systems, Prikladnaja Matematica I Mekhanica (PMM USSR), 38, pp. 425–429, 1974.

[20] T.-L. Yang and R. Rosenberg, On forced vibrations of a particle in the plane, International Journal of Non-Linear Mechanics, 3, pp. 47–63, 1968.

[21] A. Vakakis and T. Caughey, A theorem on the exact nonsimilar steady state motions of a nonlinear oscillator, Journal of Applied Mechanics, 59, pp. 418–424, 1992.

[22] K.V. Avramov, Analysis of forced vibrations by nonlinear modes// Nonlinear Dynamics. Vol. 53. pp. 117-127, 2008.

[23] Yu. Mikhlin and B. Morgunov, Normal vibrations in near-conservative self-exited and viscoelastic nonlinear systems, Nonlinear Dynamics, 25, pp. 33-48, 2001.

[24] K.V. Avramov, Nonlinear modes of parametric vibrations and their applications to beams dynamics// Journal of Sound and Vibrations. Vol. 322, No 3, pp. 476-489, 2009.

[25] A. Vakakis, L. Manevitch, Y. Mikhlin, V. Pilipchuk and A. Zevin, Normal modes and localization in nonlinear systems, Wiley, New-York. 552 pps, 1996.

[26] A.K. Mishra and M. S. Singh, The normal modes of nonlinear symmetric systems by group representation theory// Int. J. Nonl. Mech. Vol. 9. pp. 463-480, 1974.

[27] G.M. Chechin, V.P. Sakhnenko, H.T. Stokes, A.D. Smith and D.M. Hatch, Non-linear normal modes for systems with discrete symmetry. Int. J. of Non-Linear Mechanics. Vol. 35, pp. 497-513, 2000.

[28] L. Zuo and A. Curnier, Non-linear real and complex modes of conewise linear systems// J. of Sound and Vib. Vol. 174. №3. pp. 289-313, 1994.

[29] V.N. Pilipchuk, Impact modes in discrete vibrating systems with rigid barriers, Int. J. of Non-Linear Mechanics. Vol. 36, pp. 999-1012, 2001.

[30] S. Belizzi and R. Bouc,. A new formulation for the existence and calculation of nonlinear normal modes//

J. of Sound Vibration. Vol. 287. № 3. pp. 545-569, 2005.

[31] M.E. King and A.F. Vakakis, An energy-based formulation for computing nonlinear normal modes in undamped continuous systems// J. of Vibration and Acoustics. Vol. 116. №3. pp. 332-340, 1993.

[32] S.W. Shaw and C. Pierre, Normal modes of vibration for non-linear continuous systems// J. of Sound and Vibration. Vol. 169. №3. pp. 319-347, 1994.

[33] A. Nayfeh and S. Nayfeh, On nonlinear modes of continuous systems, J. of Vibration and Acoustics, 116, pp. 129–136, 1994.

[34] A.F. Vakakis, L.I. Manevitch, O. Gendelman, and L. Bergman, Dynamics of linear discrete systems connected to local, essentially non-linear attachments. J. of Sound and Vibration, 264, pp. 559–577, 2003.

[35] O.V. Gendelman, Bifurcations of Nonlinear Normal Modes of Linear Oscillator with Strongly Nonlinear Damped Attachment, Nonlinear Dynamics, V.37, N.2, pp. 115-128, 2004.

[36] K.V. Avramov and Yu.V. Mikhlin,. Snap- through truss as a vibration absorber, J. of Vibration and Control. V. 10. pp. 291-308, 2004.

[37] K.V. Avramov and Yu.V. Mikhlin, Forced oscillations of a system containing a snap- through truss, close to its equilibrium position, Nonlinear Dynamics, V. 35, pp. 361- 379, 2004.

[38] K.V. Avramov and Yu.V. Mikhlin, Snap-through truss as an absorber of forced oscillations, J. of Sound and Vibration, Vol. 29, P. 705-722, 2006.

[39] Yu. V. Mikhlin and S.N. Reshetnikova, Dynamical interaction of an elastic system and essentially nonlinear absorber, J. of Sound and Vibration, Vol. 283, pp. 91-120, 2005.

[40] K.V. Avramov, Yu.V. Mikhlin and E. Kurilov, 2007, Asymptotic analysis of nonlinear dynamics of simply supported cylindrical shells, Nonlinear Dynamics, Vol. 47, pp. 331-352, 2007.

[41] I.D. Breslavsky, K.V. Avramov, Yu. V. Mikhlin and R. Kochurov, Nonlinear modes of snap-through motions of a shallow arch, J. of Sound and Vibration, Vol. 311, pp. 297-313, 2008.

[42] M. Rauscher, Steady oscillations of systems with nonlinear and unsymmetric elasticity, J. of Applied Mechanics, 5, A169-A177, 1938.

[43] R.S. Haxton and A.D.S. Barr, The autoparametric vibration absorber, J. of Engineering for Industry, 94, pp. 119-225, 1972.

[44] S. Natsiavas, Steady state oscillations and stability of non-linear dynamic vibration absorber, J. of Sound and Vibration, Vol. 156, No 22, pp. 227-245. 1992.

[45] S. Natsiavas, Vibration absorbers for a class of self-excited mechanical systems, Trans**.** of the ASME. J. of Applied Mechanics, 60, pp. 382-387. 1993**.**

[46] Vibrations in engineering (Frolov K.V., editor), Moscow, Mashinostroenie, 1995 (in Russian).

[47] A.G. Haddow and S.W. Shaw, An experimental study of torsional vibration absorber, Proc. of the ASME Design Engineering Technical Conference, USA, pp. 1-7, 2001.

[48] O.V. Gendelman, A.F. Vakakis, L.I. Manevitch and R. McCloskey, Energy Pumping in Nonlinear Mechanical Oscillators I: Dynamics of the Underlying Hamiltonian System, J. of Applied Mechanics, 68, pp. 34-41, 2001; A.F. Vakakis and O.V. Gendelman, Energy Pumping in Nonlinear Mechanical Oscillators II: Resonance Capture, ibid, pp. 42-48, 2001.

[49] A.F.Vakakis, L.I.Manevitch, O.Gendelman, and L.Bergman, Dynamics of linear discrete systems connected to local, essentially non-linear attachments. J. of Sound and Vibration, 264, pp. 559–577, 2003.

[50] M. P. Karyeaclis and T. K. Caughey, Stability of a semi-active impact damper, Trans. of the ASME. J. of Applied Mechanics 56, pp. 926-940. 1989.

[51] S. Aoki and T. Watanabe, An investigation of an impact vibration absorber with hysteretic damping, Proc. of the Second International Conference on Motion and Vibration Control, Yokohama, Japan., pp. 468-474, 1994.

[52] J. Shaw, S.W. Shaw and A.G., Haddow, On the response of the non-linear vibration absorber. Int. J. of Non-Linear Mechanics, Vol.24, N 4, pp. 281-293, 1989.

[54] P.M. Alabuchev, A.A. Gritchin and L.I. Kim, L.I., Vibration insulation systems with quasi-zero stiffness, Mashinostroenie, Leningrad, 1986 (in Russian).

[55] A.H. Nayfeh and D.T. Mook, Nonlinear Oscillations. New York: Wiley, 1979.

[56] B. Budiansky and J.W. Hutchinson, Dynamic buckling of imperfect-sensitive structures, Appl. Mechanics: Springer Verlag, pp. 636 – 657, 1966.

[57] V.D. Kubenko V.D. and P.S. Kovalchuk, Influence of geometrical initial imperfections to vibrations and dynamical instability of elastic shells. Int. Appl. Mech. Vol. 40, No 8, pp. 26 – 65, 2004.

[58] V.V. Bolotin, Non-conservative Problems of the Theory of Elastic Stability. New York, MacMillan, 1963.

[59] M.D. Olson and Y.C. Fung, Comparing theory and experiment for the supersonic flutter of circular shell, AIAA J., No 10, pp. 1849-1856, 1967.

[60] Thin- Shell Structures, Theory, Experiment and Design (Y.C. Fung and E.E. Sechler, eds), Prentice-Hall, Englewood Cliff, New Jersey, 1974.

[61] M. Amabili and F. Pellicano, Nonlinear supersonic flatter of circular cylindrical shells, AIAA Journal. Vol. 39, N 4, pp. 564-572, 2001.

[62] G.W. Barr and R.O. Stearman, Aeroelastic stability characteristics of cylindrical shells considering

imperfections and edge constraint, AIAA J. N7, pp. 912 – 919, 1967.

[63] Kubenko, V.D., Koval'shuk, P.S. and T.S. Krasnopolskaya, Nonlinear Interaction of Modes of Bending Shell Oscillations. Kiev: Naukova Dumka (in Russian), 1984.

[64] R.A. Williams, Automotive active suspensions, Part 1: Basic principles, Proceedings of the Institute of Mechanical Engineers, J. of Automobile Engineering, Part D 211, pp. 415–426, 1997.

[65] M. Haroon, D. E. Adams, Y. WahLuk, A. A. Ferri, A time and frequency domain approach for identifying nonlinear mechanical system models in the absence of an input measurement, J. of Sound and Vibration 283, pp. 1137–1155, 2005.

[66] D. Hrovat, Optimal suspension performance for 2-D vehicle models, J. of Sound and Vibration, Vol. 146, No l, pp. 93-110, 1991.

[67] J. Marzbanrad, G. Ahmadi, H. Zohoor, Y. Hojjat, Stochastic optimal preview control of a vehicle suspension, J. of Sound and Vibration 275, pp. 973–990, 2004.

[68] J.Y. Wong, The theory of ground vehicles. 2nd ed. New York: John Wiley, 1993.

[69] Hyo-Jun Kim, Hyun Seok Yang, Young-Pil Park, Improving the vehicle performance with active suspension using road-sensing algorithm. Computers and Structures 80, pp. 1569–1577, 2002.

[70] V. Pilipchuk, N. Moshchuk, Shih-Ken Chen, Vehicle state estimation using dynamic 7 degree-of-freedom observer. Proc. of IMECE2006, Chicago, USA, Nov. 2006, 5-10, pp. 1-8, 2006.

[71] Q. Zhu, M. Ishitobi, Chaos and bifurcations in a nonlinear vehicle model. // Journal of Sound and Vibration, 275, pp. 1136-1146, 2004.