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Forced Oscillations of a System, Containing a Snap-Through Truss, Close to Its Equilibrium Position

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Abstract. The nonlinear dynamics of a two-degree-of-freedom mechanical system is considered. This system consists of a linear oscillator under the action of a time-periodic force and a snap-through truss, which acts as an absorber of the forced oscillations of the linear main system. The forced oscillations of the snap-through truss close to its equilibrium position are analyzed by the multiple scales method.

Key words: multiple scales method, parametric resonance, snap-through truss, vibrations absorption

1. Introduction

Many efforts were made to study dynamical systems with snap-through motions, which is due to the importance of these problems in engineering [1]. Mises [2] was the first, who studied a static of the snap-through system. It seems that the first study of the snap-through truss dynamics was performed by Stoker [3]. Nachbar and Huang [4] considered the snap-through truss under the action of a static force. They formulated a criterion of the snap-through motions arising, which is based on the analysis of the dynamical system on the plane. The multiple scales method was used to analyze the snap-through motions by Reissand Matkowsky [5]. The snap-through shallow truss without dissipation is considered in the paper [6]. It is assumed, that the excitation frequency has a great value. Therefore, the system oscillations are presented as the sum of slow and fast motions. A significant contribution to the theory of the snap-through motions was made by Holmes [7]. He suggested the asymptotic Melnikov method to study these motions. Clemeus and Wauer [8] suggested a design of the snap-through truss consisting of two rods. They used the harmonic balance method to study the snap-through motions. Cook and Simiu [9] studied experimentally the periodic and chaotic snap- through motions of the Stoker column. They described the design of the Stoker column, which was used in experimental studies. The periodic orbits bifurcations, which precede the snap-through motions, were analyzed in [10]. Many-mode approximations for snap-through elastic systems are considered in [11].

Besides the analysis of the snap-through dynamics, there is one more interesting problem: an analysis of oscillations close to equilibrium positions. Bifurcations and instability of such oscillations lead to snap-through. Holmes and Holmes [12] considered such motions in the one-degree-of-freedom system. They studied the period-doubling bifurcation of these motions. Virgin [13] analyzed the small linear oscillations close to equilibriums of systems with snap-through motions. Dynamics of different oscillators close to equilibriums are studied in [14].

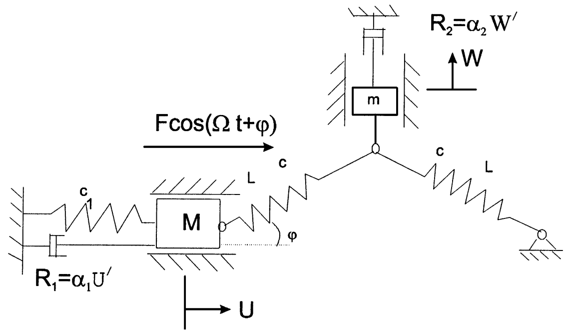

Figure 1. Mechanical system under consideration.

In this paper the forced oscillations of the two-degree-of-freedom system (Figure 1) are investigated. The mass M is the main mass, as the forced oscillations of this mass must be quenched. The snap-through truss is attached to this mass to absorb the oscillations. The analysis of quenching of this system free oscillations was reported in the previous paper of the authors [15]. We showed that the vibration absorption is presented by stable motions with small amplitudes of the main system and the snap-through motions of the truss. In this case, the free oscillations energy of the main linear system is pumped over the snap-through truss oscillation energy. Now the following problem has risen naturally. Let us assume that the periodic force acts on the main mass. Can these forced oscillations be quenched by means of the snap-through truss? The effective quenching takes place if the main mass has small amplitudes and the absorber has the snap- through motions. However, besides such modes the other motions can be observed. One such motion is the oscillations with small amplitudes of the snap-through truss close to the stable equilibrium position. These motions are studied in this paper. The instability regions of such motions are obtained here. In these regions the oscillations are increased and the absorber can fall into the snap-through motions, which are effective for the absorption of oscillations. The resonance motion regions, where the snap- through truss has moderately large amplitudes of oscillations close to one of the stable equilibrium positions, are determined. The authors of this paper show by means of the method of nonlinear normal modes [16], that the forced stable snap-through motions of the absorber exist in the wide range of the forcing frequency. Therefore, the parameters of absorber must be chosen so that resonance motions close to the stable equilibrium position are unstable and all motions are attracted to the snap-through motions.

A typical engineering situation is that the mass and stiffness of the absorber are significantly smaller than the corresponding parameters of the main elastic system, which is used in this paper.

In the vicinity of the resonances, which are considered in this paper, the analysis of forced oscillations is reduced to the two-degree-of-freedom system with parametric excitation. Let us enumerate some works devoted to such systems analysis. The system dynamics with a parametric excitation is considered in the book [17]. Parametrically excited system with not simple internal resonance 1:1 is treated in [18]. Nayfeh [19] investigated a two-degree-of-freedom system with quadratic non- linearity in the case of the main parametric resonance and internal resonance 2:1. Bryant and Miles [20] studied the dynamics of the periodically forced damped pendulum by the method of harmonic balance. The parametric oscillations of the double pendulum are studied by perturbation methods in [21]. Streit et al. [22] analyzed the parametric oscillations of two-degree-of-freedom system, which is simulated by a manipulator dynamics. Bolotin [23] studied the Rayleigh equation with parametric terms by the harmonic balance method.

2. Problem Formulation

Forced oscillations of the snap-through truss (Figure 1) are assumed to occur close to the stable equilibrium position. Therefore, the oscillations are small. The system motions are defined by two generalized coordinates $U,\ W$. The damping forces $R_1 = \bar{\alpha}_1 \dot{U}; R_2 = \bar{\alpha}_2 \dot{W}$ act on both masses. The equations of motions have the following form:

$$M\ddot{U} + \bar{c}_1 U + 2\bar{c}\left[U - L\cos\varphi + L\frac{L\cos\varphi - U}{\sqrt{L^2 + 2L(W\sin\varphi - U\cos\varphi) + U^2 + W^2}}\right]$$
$$= F\cos(\Omega t + \bar{\varphi}) - \bar{\alpha}_1 \dot{U}, \tag{1}$$
$$m\ddot{W} + 2\bar{c}\left[L\sin\varphi + W - \frac{L(L\sin\varphi + W)}{\sqrt{L^2 + 2L(W\sin\varphi - U\cos\varphi) + U^2 + W^2}}\right] = -\bar{\alpha}_2 \dot{W}. \tag{2}$$

Dimensionless variables and parameters are introduced:

$$\frac{U}{L} = u;\quad \frac{W}{L} = w;\quad \omega_0^2 = \frac{\bar{c}_1}{M};\quad t = \frac{\tau}{\omega_0};\quad \frac{\Omega}{\omega_0} = \omega;\quad \gamma = \frac{\bar{c}}{\bar{c}_1};\quad \mu = \frac{m}{M};$$
$$\frac{F}{\bar{c}_1 L} = \varepsilon^k f,\quad \varepsilon\alpha_1 = \frac{\bar{\alpha}_1\omega_0}{\bar{c}_1},\quad \varepsilon\alpha_2 = \frac{\bar{\alpha}_2\omega_0}{\bar{c}_1}, \tag{3}$$

where $\varepsilon \ll 1$. Then the equations of motions with respect to the dimensionless variables can be obtained in the form:

$$\ddot{u} + u + 2\gamma(u - c)\left[1 - \frac{1}{\sqrt{1 + 2(ws - uc) + u^2 + w^2}}\right] = \varepsilon^k f\cos(\omega t + \bar{\varphi}) - \varepsilon\alpha_1\dot{u},$$
$$\mu\ddot{w} + 2\gamma(s + w)\left[1 - \frac{1}{\sqrt{1 + 2(ws - uc) + u^2 + w^2}}\right] = -\varepsilon\alpha_2\dot{w}, \tag{4}$$

where $s = \sin\varphi; c = \cos\varphi$. As small oscillations are considered, the change of variables $u \mapsto \varepsilon u,\ w \mapsto \varepsilon w$ is introduced. Using expansion in terms of u, w, the following dynamical system is derived:

$$\ddot{u} + u(1 + 2\gamma c^2) - \gamma 2csw = \varepsilon\gamma\lfloor 3cs^2u^2 + (1 - 3s^2)cw^2 + 2s(3c^2 - 1)wu\rfloor$$
$$+ \varepsilon^{k-1} f\cos(\omega t + \varphi) - \varepsilon\alpha_1\dot{u}, \tag{5}$$
$$\ddot{w} + 2p^2s^2w - 2p^2scu = \varepsilon 2p^2\left[u^2\frac{s}{2}(3c^2 - 1) - w^2\frac{3}{2}sc^2 + wuc(1 - 3s^2)\right] - \varepsilon\alpha_2\dot{w}. \tag{6}$$

The system (5, 6) for $\varepsilon = 0$ is linear with the fundamental frequencies ν_1 and ν_2:

$$2\nu_{1,2}^2 = 1 + 2p^2s^2 + 2\gamma \mp \sqrt{(1 + 2p^2s^2 + 2\gamma)^2 - 8p^2s^2}. \tag{7}$$

Note that the relation of the generalized coordinates u, w to the normal coordinates x, y of the system (5, 6) for $\varepsilon = 0$ is given in Appendix 1. The Equations (5, 6) with respect to x, y have the following form:

$$\ddot{x} + \nu_1^2 x = \varepsilon\lfloor \delta_1 \dot{y} - \delta_2 \dot{x} + A_1 x^2 + A_2 y^2 + A_3 xy + \varepsilon^{k-2} f \cos(\omega t + \varphi)\rfloor,$$
$$\ddot{y} + \nu_2^2 y = \varepsilon\left[\dot{x}\frac{\delta_1}{\mu} - \dot{y}\delta_3 + B_1 x^2 + B_2 y^2 + B_3 xy + \varepsilon^{k-2}\tilde{\alpha} f \cos(\omega t + \varphi)\right]. \quad (8)$$

Formulae for the system (8) parameters are given in Appendix 1.

3. Resonance $\nu_1 \approx \nu_2$ and $\omega \approx 2\,\nu_1$

The internal resonance 1:1 is defined in the following way:

$$\nu_1 = \nu_2 + \varepsilon\sigma, \quad (9)$$

where σ is the detuning parameter. As follows from (7), this resonance occurs if the system parameters satisfy the equation:

$$(1 + 2\gamma)^2 = 4p^2 s^2 (1 - 2\gamma). \quad (10)$$

As follows from (10), the parameter γ meets the following inequality: $\frac{1}{2} > \gamma$. The disturbance frequency satisfies the following resonance condition:

$$\omega = 2\nu_1 + \varepsilon\lambda. \quad (11)$$

Note that the resonance (11) occurs at $k = 1$. The change of variables is used:

$$x = \Lambda\cos(\omega t + \varphi) + \xi; \quad y = \tilde{\alpha}\Lambda\cos(\omega t + \varphi) + \eta. \quad (12)$$

Then the system (8) has the following form with respect to the new variables ξ and η:

$$\ddot{\xi} + \nu_1^2 \xi = \varepsilon\Big\{\delta_1\dot{\eta} - \delta_2\dot{\xi} + F_A^{(1)}\cos(2\omega t + 2\varphi) + A_1\xi^2 + A_2\eta^2 + A_3\xi\eta + D_A^{(2)}\xi\cos(\omega t + \varphi) + D_A^{(3)}\eta\cos(\omega t + \varphi) + \cdots\Big\} \quad (13)$$

$$\ddot{\eta} + \nu_1^2 \eta = \varepsilon\Big\{2\sigma\nu_1\eta + \dot{\xi}\frac{\delta_1}{\mu} - \dot{\eta}\delta_3 + F_B^{(1)}\cos(2\omega t + 2\varphi) + B_1\xi^2 + B_2\eta^2 + B_3\xi\eta + D_B^{(2)}\xi\cos(\omega t + \varphi) + D_B^{(3)}\eta\cos(\omega t + \varphi) + \cdots\Big\} \quad (14)$$

Appendix 1 contains the values $F_A^{(1)}$, $D_A^{(2)}$, $D_A^{(3)}$, $F_B^{(1)}$, $D_B^{(2)}$, $D_B^{(3)}$. The unessential terms for the analysis are not presented in system (13, 14). The multiple scales method [17] is used to analyze system (13, 14). Then the solutions are presented in the form:

$$\xi = \xi_0(T_0, T_1, \ldots) + \varepsilon\xi_1(T_0, T_1, \ldots) + \ldots; \eta = \eta_0(T_0, T_1, \ldots) + \varepsilon\eta_1(T_0, T_1, \ldots) + \ldots, \quad (15)$$

where $T_0 = t; T_1 = \varepsilon t$. If (15) is substituted into the system (13, 14) and some algebra is carried out, the following equations are derived:

$$\frac{\partial^2 \xi_0}{\partial T_0^2} + \nu_1^2 \xi_0 = 0; \quad \frac{\partial^2 \eta_0}{\partial T_0^2} + \nu_1^2 \eta_0 = 0; \tag{16}$$

$$\frac{\partial^2 \xi_1}{\partial T_0^2} + \nu_1^2 \xi_1 = -2\frac{\partial^2 \xi_0}{\partial T_0 \partial T_1} + \delta_1 \frac{\partial \eta_0}{\partial T_0} - \delta_2 \frac{\partial \xi_0}{\partial T_0} + F_{\mathrm{A}}^{(1)} \cos(2\omega t + 2\varphi) + A_1 \xi_0^2 + A_2 \eta_0^2 + A_3 \xi_0 \eta_0 + D_{\mathrm{A}}^{(2)} \xi_0 \cos(\omega t + \varphi) + D_{\mathrm{A}}^{(3)} \eta_0 \cos(\omega t + \varphi) + \ldots; \tag{17}$$

$$\frac{\partial^2 \eta_1}{\partial T_0^2} + \nu_1^2 \eta_1 = -2\frac{\partial^2 \eta_0}{\partial T_0 \partial T_1} + 2\sigma \nu_1 \eta_0 + \frac{\delta_1}{\mu} \frac{\partial \xi_0}{\partial T_0} - \delta_3 \frac{\partial \eta_0}{\partial T_0} + F_{\mathrm{B}}^{(1)} \cos(2\omega t + 2\varphi) + B_1 \xi_0^2 + B_2 \eta_0^2 + B_3 \xi_0 \eta_0 + D_{\mathrm{B}}^{(2)} \xi_0 \cos(\omega t + \varphi) + D_{\mathrm{B}}^{(3)} \eta_0 \cos(\omega t + \varphi) + \ldots. \tag{18}$$

Solutions of the Equations (16) are well known.

$$\xi_0 = A_1 \exp(i\nu_1 T_0) + \bar{A}_1 \exp(-i\nu_1 T_0); \quad \eta_0 = A_2 \exp(i\nu_1 T_0) + \bar{A}_2 \exp(-i\nu_1 T_0),$$

where $A_1 = \frac{a_1}{2} \exp(i\psi_1); A_2 = \frac{a_2}{2} \exp(i\psi_2)$. Using the condition of absence of secular terms, the following modulation equations can be obtained:

$$\begin{aligned}
& a_1' - \delta_1 \frac{a_2}{2} \cos(\psi_2 - \psi_1) + \frac{\delta_2}{2} a_1 - \frac{D_{\mathrm{A}}^{(2)} a_1}{4\nu_1} \sin(\lambda T_1 + \varphi - 2\psi_1) \\
& \quad - \frac{D_{\mathrm{A}}^{(3)} a_2}{4\nu_1} \sin(\lambda T_1 + \varphi - \psi_2 - \psi_1) = 0; \\
& \psi_1' a_1 - \frac{\delta_1 a_2}{2} \sin(\psi_2 - \psi_1) + \frac{D_{\mathrm{A}}^{(2)} a_1}{4\nu_1} \cos(\lambda T_1 + \varphi - 2\psi_1) + \frac{D_{\mathrm{A}}^{(3)} a_2}{4\nu_1} \\
& \quad \times \cos(\lambda T_1 - \varphi - \psi_2 - \psi_1) = 0; \\
& a_2' - \frac{\delta_1 a_1}{2\mu} \cos(\psi_1 - \psi_2) + \frac{\delta_3 a_2}{2} - \frac{D_{\mathrm{B}}^{(2)} a_1}{4\nu_1} \sin(\lambda T_1 + \varphi - \psi_1 - \psi_2) \\
& \quad - \frac{D_{\mathrm{B}}^{(3)} a_2}{4\nu_1} \sin(\lambda T_1 + \varphi - 2\psi_2) = 0; \\
& \psi_2' a_2 + \sigma a_2 - \frac{\delta_1 a_1}{2\mu} \sin(\psi_1 - \psi_2) + \frac{D_{\mathrm{B}}^{(2)} a_1}{4\nu_1} \cos(\lambda T_1 + \varphi - \psi_1 - \psi_2) \\
& \quad + \frac{D_{\mathrm{B}}^{(3)} a_2}{4\nu_1} \cos(\lambda T_1 + \varphi - 2\psi_2) = 0.
\end{aligned} \tag{19}$$

3.1. Analysis of the Case $\delta_1 = \delta_2 = \delta_3 = 0$

In this section the resonance $\nu_1 \approx \nu_2$ and $\omega \approx 2\nu_1$ is considered in the system (8) at $\delta_1 = \delta_2 = \delta_3 = 0$. The change of variables is introduced:

$$(x_1, y_1, x_2, y_2) = \left\{ a_1 \sin\left(\frac{\lambda}{2} T_1 - \psi_1\right);\ a_1 \cos\left(\frac{\lambda}{2} T_1 - \psi_1\right);\ a_2 \sin\left(\frac{\lambda}{2} T_1 - \psi_2\right);\ a_2 \cos\left(\frac{\lambda}{2} T_1 - \psi_2\right) \right\}. \tag{20}$$

Then the system of modulation Equations (19) has the following form:

$$\begin{aligned} \dot{x}_1 &= \left(\frac{D_{\text{A}}^{(2)}}{4\nu_1} + \frac{\lambda}{2}\right) y_1 + \frac{D_{\text{A}}^{(3)}}{4\nu_1} y_2; \; \dot{y}_1 = \left(\frac{D_{\text{A}}^{(2)}}{4\nu_1} - \frac{\lambda}{2}\right) x_1 + \frac{D_{\text{A}}^{(3)}}{4\nu_1} x_2; \\ \dot{x}_2 &= \frac{D_{\text{B}}^{(2)}}{4\nu_1} y_1 + \left(\frac{\lambda}{2} + \sigma + \frac{D_{\text{B}}^{(3)}}{4\nu_1}\right) y_2; \; \dot{y}_2 = \frac{D_{\text{B}}^{(2)}}{4\nu_1} x_1 + \left(-\frac{\lambda}{2} - \sigma + \frac{D_{\text{B}}^{(3)}}{4\nu_1}\right) x_2; \end{aligned} \tag{21}$$

The following equations connect the variables of the system (21) to ξ and η:

$$\begin{aligned} \xi(t) &= y_1 \cos\left(\frac{\omega}{2} t\right) + x_1 \sin\left(\frac{\omega}{2} t\right) + O(\varepsilon), \\ \eta(t) &= y_2 \cos\left(\frac{\omega}{2} t\right) + x_2 \sin\left(\frac{\omega}{2} t\right) + O(\varepsilon). \end{aligned} \tag{22}$$

The vibrations mode of the mechanical system follows from the solutions of the system (21). If trivial solutions of system (21) are unstable, the oscillations of the snap-through truss increases, and the steady motion is the snap-through, which is effective to absorb the oscillations of mass M.

Let us obtain the domains of instability of the trivial solutions of the system (21). This system's solutions have the following form:

$$(x_1, y_1, x_2, y_2) = \exp(\chi t)\,(C_1, C_2, C_3, C_4). \tag{23}$$

The value χ is defined from the following biquadratic equation:

$$\chi^4 + a\chi^2 + g = 0, \tag{24}$$

where

$$\begin{aligned} a &= -\frac{1}{16\nu_1^2}\left(D_{\text{B}}^{(3)^2} + 2D_{\text{A}}^{(3)} D_{\text{B}}^{(2)} + D_{\text{A}}^{(2)^2}\right) + \frac{\lambda^2}{4} + \left(\frac{\lambda}{2} + \sigma\right)^2. \\ g &= -\frac{D_{\text{A}}^{(3)} D_{\text{B}}^{(2)}}{16\nu_1^2}\left[\frac{D_{\text{A}}^{(2)} D_{\text{B}}^{(3)}}{8\nu_1^2} + \frac{\lambda}{2}(2\sigma + \lambda)\right] + \frac{D_{\text{A}}^{(3)^2} D_{\text{B}}^{(2)^2}}{16^2 \nu_1^4} - \left(\frac{D_{\text{A}}^{(2)^2}}{16\nu_1^2} - \frac{\lambda^2}{4}\right) \\ &\quad \times \left[-\frac{D_{\text{B}}^{(3)^2}}{16\nu_1^2} + \left(\frac{\lambda}{2} + \sigma\right)^2\right]. \end{aligned}$$

Note, that the equation of the stable/unstable trivial solutions boundary has the form $\chi = 0$. This equation can be written as

$$\begin{aligned} &-r\left[2\sqrt{nl} + \lambda\left(\sigma + \frac{\lambda}{2}\right)\right] + \frac{r^2}{4} - 4\left(n - \frac{\lambda^2}{4}\right)\left[-l + \left(\frac{\lambda}{2} + \sigma\right)^2\right] \\ &\quad = \left[-l - n - \frac{r}{2} + \frac{\lambda^2}{4} + \left(\frac{\lambda}{2} + \sigma\right)^2\right]^2. \end{aligned} \tag{25}$$

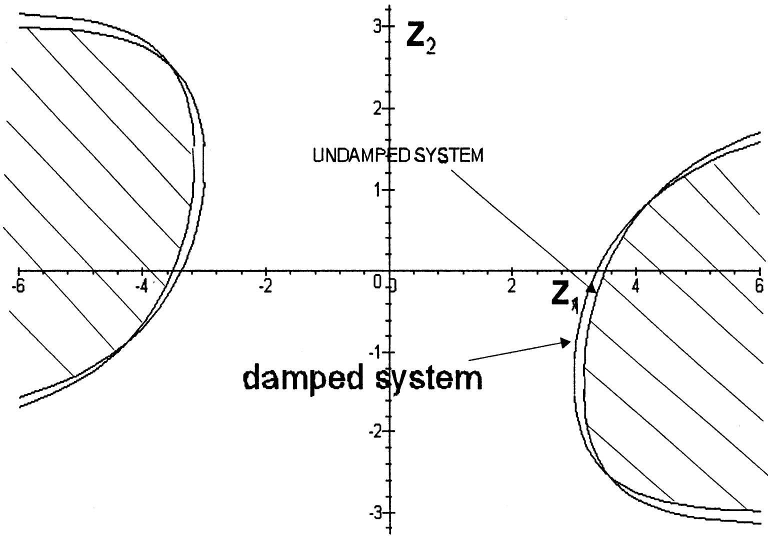

Figure 2. Boundaries of the stable/unstable solutions in the case $\delta_1 = \delta_2 = \delta_3 = 0$ with the following system parameters: $\gamma = 0.1$; $p = 3$; $f = 1$; $\varphi = 0.22$; $s = 0.22$; $c = 0.97$. The regions of the unstable oscillations is shaded.

Appendix 1 contains the expressions for r, l, n. Let us analyze the regions of instability in the plane (λ, σ). Equation (25) has the following form in the new variables $(z_1, z_2) = (\sigma, \lambda + \sigma)$:

$$r\left[z_1^2 - (\sqrt{l} + \sqrt{n})^2\right] - [z_1 z_2 - l + n]^2 = 0. \tag{26}$$

The boundaries between the stable/unstable solutions have the following form:

$$z_2 = \frac{l - n}{z_1} \pm \sqrt{r}\sqrt{1 - \frac{(\sqrt{l} + \sqrt{n})^2}{z_1^2}}. \tag{27}$$

As follows from (27), the next relation holds:

$$\lim_{z_1 \to \pm\infty} z_2 = \pm\sqrt{r}. \tag{28}$$

The following system parameters are used to study the boundary between the stable and unstable oscillations: $\gamma = 0.1$, $p = 3$, $f = 1$. In this case, the resonance $\nu_1 \approx \nu_2$ occurs at $\varphi = 0.22, s = 0.22, c = 0.97$. As the result of the calculations, the following values are obtained: $n = 5.13, l = 0.78, r = 8.04$. Figure 2 shows the considered boundary. The regions of the unstable oscillations are shaded.

3.2. Analysis of the Case $\delta_1 \neq 0, \delta_2 \neq 0, \delta_3 \neq 0$

In this section, the resonance $\nu_1 \approx \nu_2$, $\omega \approx 2\nu_1$ is considered in the system (8) at $\delta_1 \neq 0, \delta_2 \neq 0, \delta_3 \neq 0$. This system behavior is described by the modulation Equations (19). The boundary between the stable and unstable solutions of these modulation equations is studied. The system (19) is rewritten with respect to (x_1, y_1, x_2, y_2) (see Equation (20)):

$$\begin{aligned}
\dot{x}_1 &= -\frac{\delta_2}{2}x_1 + \frac{\delta_1}{2}x_2 + \frac{D_A^{(3)}}{4\nu_1}y_2 + \left(\frac{D_A^{(2)}}{4\nu_1} + \frac{\lambda}{2}\right)y_1;\\
\dot{y}_1 &= \left(\frac{D_A^{(2)}}{4\nu_1} - \frac{\lambda}{2}\right)x_1 - \frac{\delta_2}{2}y_1 + \frac{D_A^{(3)}}{4\nu_1}x_2 + \frac{\delta_1}{2}y_2;\\
\dot{x}_2 &= \frac{\delta_1}{2\mu}x_1 + \frac{D_B^{(2)}}{4\nu_1}y_1 - \frac{\delta_3}{2}x_2 + \left(\frac{D_B^{(3)}}{4\nu_1} + \frac{\lambda}{2} + \sigma\right)y_2;\\
\dot{y}_2 &= \frac{D_B^{(2)}}{4\nu_1}x_1 + \frac{\delta_1}{2\mu}y_1 + \left(\frac{D_B^{(3)}}{4\nu_1} - \frac{\lambda}{2} - \sigma\right)x_2 - \frac{\delta_3}{2}y_2.
\end{aligned} \tag{29}$$

The solutions of system (29) are presented in the form (23). Then the value $\tilde{\chi}$ is defined from the equation:

$$\begin{vmatrix}
-\frac{\delta_2}{2} - \tilde{\chi} & \frac{D_A^{(2)}}{4\nu_1} + \frac{\lambda}{2} & \frac{\delta_1}{2} & \frac{D_A^{(3)}}{4\nu_1}\\
\frac{D_A^{(2)}}{4\nu_1} - \frac{\lambda}{2} & -\frac{\delta_2}{2} - \tilde{\chi} & \frac{D_A^{(3)}}{4\nu_1} & \frac{\delta_1}{2}\\
\frac{\delta_1}{2\mu} & \frac{D_B^{(2)}}{4\nu_1} & -\frac{\delta_3}{2} - \tilde{\chi} & \frac{D_B^{(3)}}{4\nu_1} + \frac{\lambda}{2} + \sigma\\
\frac{D_B^{(2)}}{4\nu_1} & \frac{\delta_1}{2\mu} & \frac{D_B^{(3)}}{4\nu_1} - \frac{\lambda}{2} - \sigma & -\frac{\delta_3}{2} - \tilde{\chi}
\end{vmatrix} = 0. \tag{30}$$

Analytical expressions for $\tilde{\chi}$ can be derived only in the case $\alpha_1 = \alpha_2 = \alpha_*$. Then $\delta_1 = 0; \delta_2 = \delta_3 = \alpha_*$.

The stability/instability of the trivial solutions boundary is derived with respect to the variables (z_1, z_2) (Equation (26)):

$$\left(z_1^2 + \alpha_*^2\right)z_2^2 - 2(l-n)z_1z_2 - (r - \alpha_*^2)z_1^2 - \bar{C} = 0, \tag{31}$$

where

$$\bar{C} = -(l-n)^2 - r(\sqrt{l} + \sqrt{n})^2 - \alpha_*^4 - 2\alpha_*^2\left(l + n + \frac{r}{2}\right).$$

This boundary can be written in the form:

$$z_2 = \frac{(l-n)z_1 \pm \sqrt{(l-n)^2z_1^2 + (z_1^2 + \alpha_*^2)[(r - \alpha_*^2)z_1^2 + \bar{C}]}}{z_1^2 + \alpha_*^2}. \tag{32}$$

At $\alpha_* = 0$ the boundary (32) has the form (27). Figure 2 shows the boundary (32) for the parameters presented in the subsection 3.1 and $\alpha_* = 1$.

4. Resonance $\nu_1 \approx \nu_2$ and $\omega \approx \nu_1$

Oscillations of the system (8) are considered in the case of the internal resonance (9) and the additional resonance condition:

$$\omega = \nu_1 + \varepsilon\alpha. \tag{33}$$

Following the multiple scales method, the oscillations of the system (8) are presented in the form: $x = x_0(T_0, T_1, \ldots) + \varepsilon x_1(T_0, T_1, \ldots) + \ldots; y = y_0(T_0, T_1, \ldots) + \varepsilon y_1(T_0, T_1, \ldots) + \ldots$. Then the following equations holds:

$$x_0 = A_1 \exp(i\nu_1 T_0) + \bar{A}_1 \exp(-i\nu_1 T_0),\ y_0 = A_2 \exp(i\nu_1 T_0) + \bar{A}_2 \exp(-i\nu_1 T_0);$$
$$A_1 = \frac{a_1}{2}\exp(i\psi_1);\ A_2 = \frac{a_2}{2}\exp(i\psi_2);$$
$$\frac{\partial^2 x_1}{\partial T_0^2} + \nu_1^2 x_1 = -2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} - \delta_2 \frac{\partial x_0}{\partial T_0} + \delta_1 \frac{\partial y_0}{\partial T_0} + A_1 x_0^2 + A_2 y_0^2 + A_3 x_0 y_0 + f\cos(\omega t + \varphi); \tag{34}$$
$$\frac{\partial^2 y_1}{\partial T_0^2} + \nu_1^2 y_1 = -2\frac{\partial^2 y_0}{\partial T_0 \partial T_1} + 2\sigma\nu_1 y_0 + \frac{\delta_1}{\mu}\frac{\partial x_0}{\partial T_0} - \delta_3 \frac{\partial y_0}{\partial T_0} + B_1 x_0^2 + B_2 y_0^2$$
$$+ B_3 x_0 y_0 + \tilde{\alpha} f \cos(\omega t + \varphi). \tag{35}$$

We derive the system of modulation equations from (34, 35):

$$\begin{aligned}
&\dot{a}_1 + \frac{\delta_2}{2}a_1 - \delta_1 \frac{a_2}{2}\cos(\psi_2 - \psi_1) - \frac{f}{2\nu_1}\sin(\alpha T_1 + \varphi - \psi_1) = 0,\\
&\dot{\psi}_1 a_1 - \delta_1 \frac{a_2}{2}\sin(\psi_2 - \psi_1) + \frac{f}{2\nu_1}\cos(\alpha T_1 + \varphi - \psi_1) = 0,\\
&\dot{a}_2 - \frac{\delta_1 a_1}{\mu 2}\cos(\psi_1 - \psi_2) + \frac{\delta_3}{2}a_2 - \frac{\tilde{\alpha} f}{2\nu_1}\sin(\alpha T_1 + \varphi - \psi_2) = 0,\\
&\dot{\psi}_2 a_2 + \sigma a_2 - \frac{\delta_1 a_1}{2\mu}\sin(\psi_1 - \psi_2) + \frac{\tilde{\alpha} f}{2\nu_1}\cos(\alpha T_1 + \varphi - \psi_2) = 0.
\end{aligned} \tag{36}$$

Authors of this paper cannot derive analytical expressions for fixed points of the system (36) in the case $(\delta_1, \delta_2, \delta_3) = \mathrm{O}(1)$. If it is assumed that $\alpha_1 = \alpha_2 = \alpha_{*,}$ the system of modulation Equations (36) has the following simplified form:

$$\begin{aligned}
&\dot{a}_1 + \alpha_* \frac{a_1}{2} - \frac{f}{2\nu_1}\sin\theta_1 = 0; \quad (\alpha - \dot{\theta}_1)a_1 + \frac{f}{2\nu_1}\cos\theta_1 = 0,\\
&\dot{a}_2 + \alpha_* \frac{a_2}{2} - \frac{\tilde{\alpha} f}{2\nu_1}\sin\theta_2 = 0; \quad (\alpha - \dot{\theta}_2)a_2 + \sigma a_2 + \frac{\tilde{\alpha} f}{2\nu_1}\cos\theta_2 = 0,
\end{aligned} \tag{37}$$

where $\theta_1 = \alpha T_1 - \varphi - \psi_1;\ \theta_2 = \alpha T_1 + \varphi - \psi_2$. From system (37), the following formulae for periodic oscillations of the system (8) can be obtained:

$$x = \frac{f}{\nu_1\sqrt{4\alpha^2 + \alpha_*^2}}\cos(\omega t + \varphi - \theta_1) + O(\varepsilon);$$
$$y = \frac{\tilde{\alpha} f}{\nu_1\sqrt{\alpha_*^2 - 4(\alpha + \sigma)^2}}\cos(\omega t + \varphi - \theta_2) + O(\varepsilon). \tag{38}$$

Let us consider the case $\alpha_1 \neq \alpha_2$. It is assumed, that the coefficients of dissipation δ_1, δ_2, δ_3 are small

$$\delta_i = \varepsilon_1 \bar{\delta}_i;\ i = 1,\ 2,\ 3, \tag{39}$$

where ε_1 is a new small parameter: $0 \ll \varepsilon \ll \varepsilon_1 \ll 1$. The asymptotic solutions for the fixed points of the system (36) are derived. The oscillations of the system (8) are described by:

$$\begin{aligned} x &= -\frac{f}{2\nu_1 \alpha} \cos(\omega t + \varphi - \theta_1) + O\left(\varepsilon_1^2\right) + O(\varepsilon); \\ y &= -\frac{\tilde{\alpha} f}{2\nu_1(\alpha + \sigma)} \cos(\omega t + \varphi - \theta_2) + O\left(\varepsilon_1^2\right) + O(\varepsilon). \end{aligned} \tag{40}$$

The frequency responses (38, 40) are similar to frequency responses of linear systems. As follows from Equations (38, 40), the maximum values of amplitudes x and y are observed at $\alpha = 0$ and $\alpha = -\sigma$, respectively. If the system gets into this resonance, the snap-through motions occur.

5. The System Analysis at Small Stiffness and Mass of the Snap-Through Truss

The small values of stiffness and mass of the snap-through truss are one of the main design conditions to the system (Figure 1), because this truss is the absorber. This is taken into account in the following asymptotic relations:

$$\mu = \varepsilon \bar{\mu}; \quad \gamma = \varepsilon \bar{\gamma}. \tag{41}$$

Then the system (5, 6) has the following form:

$$\begin{aligned} &\ddot{u} + u - \varepsilon 2\bar{\gamma} c(sw - cu) = \varepsilon^{k-1} f \cos(\omega t + \varphi) - \varepsilon \alpha_1 \dot{u}, \\ &\ddot{w} + 2p^2(-scu + s^2 w) + \varepsilon 2p^2 \left[\frac{3}{2} sc^2 w^2 - wu(3c^3 - 2c) - u^2\left(\frac{3}{2}c^2 - \frac{1}{2}\right)s\right] = -\varepsilon \alpha_2 \dot{w}. \end{aligned} \tag{42}$$

The system (42) is rewritten with respect to the normal coordinates (x, y). The (x, y)–(u, w) relation is presented in Appendix 2. Then the Equations (42) have the following form:

$$\begin{aligned} &\ddot{x} + x + \varepsilon\left[\alpha_1 \dot{x} - 2\bar{\gamma} c(\beta x + sy)\right] = \varepsilon^{k-1} f \cos(\omega t + \varphi), \\ &\ddot{y} + 2p^2 s^2 y + \varepsilon\left[\alpha(\alpha_2 - \alpha_1)\dot{x} + \alpha_2 \dot{y} + 4p^2 sc\bar{\gamma}\beta(\beta x + sy) + \lambda p^2 x^2 + 3sc^2 p^2 y^2 + \Lambda p^2 xy\right] \\ &\quad = -\varepsilon^{k-1} \alpha f \cos(\omega t + \varphi). \end{aligned} \tag{43}$$

The parameters α, β, Λ, λ are also given in Appendix 2.

5.1 The Resonance $\omega \approx 2\sqrt{2}ps$

In this section the system (43) oscillations are analyzed at

$$\omega = 2\sqrt{2}ps + \varepsilon \delta_1, \tag{44}$$

where δ_1 is the detuning parameter. This resonance occurs at $k = 1$. Then the following change of variables is used:

$$x = A\cos(\omega t + \varphi) + \xi(t); \quad y = B\cos(\omega t + \varphi) + \eta(t), \tag{45}$$

where $A = \frac{f}{1-\omega^2}$; $B = \frac{\alpha f}{\omega^2 - 2p^2 s^2}$. In this case the system (43) with respect to ξ and η has the following form:

$$\begin{aligned}
&\ddot{\xi} + \xi + \varepsilon[\alpha_1\dot{\xi} - 2\bar{\gamma}c(\beta\xi + s\eta) + \cdots] = 0;\\
&\ddot{\eta} + 2p^2s^2\eta + \varepsilon[\alpha(\alpha_2 - \alpha_1)\dot{\xi} + \alpha_2\dot{\eta} + 4p^2sc\bar{\gamma}\beta(\beta\xi + s\eta) + \lambda p^2\xi^2 + 3sc^2p^2\eta^2\\
&\quad + \Lambda p^2\xi\eta + G_1\cos(\omega t + \varphi)\xi + G_2\cos(\omega t + \varphi)\eta + \cdots] = 0.
\end{aligned} \tag{46}$$

The terms, which are not essential for this analysis are not presented in the system (46). The parameters G_1 and G_2 are given in Appendix 2. The solutions of the system (46) are presented as

$$\xi = \xi_0 + \varepsilon\xi_1 + \cdots; \quad \eta = \eta_0 + \varepsilon\eta_1 + \cdots; \tag{47}$$

$$\xi_0 = A_1\exp(iT_0) + \bar{A}_1\exp(-iT_0); \quad \eta_0 = A_2\exp(i\sqrt{2}psT_0) + \bar{A}_2\exp(-i\sqrt{2}psT_0); \tag{48}$$

$$A_1 = \frac{a_1}{2}\exp(i\theta_1); \quad A_2 = \frac{a_2}{2}\exp(i\theta_2);$$

$$\begin{aligned}
\frac{\partial^2\eta_1}{\partial T_0^2} + 2p^2s^2\eta_1 = {}& -2\frac{\partial^2\eta_0}{\partial T_0\partial T_1} - \alpha(\alpha_2 - \alpha_1)\frac{\partial\xi_0}{\partial T_0} - \alpha_2\frac{\partial\eta_0}{\partial T_0} - 4p^2sc\bar{\gamma}\beta(\beta\xi_0 + s\eta_0) - \lambda p^2\xi_0^2\\
& - 3sc^2p^2\eta_0^2 - \Lambda p^2\xi_0\eta_0 - G_1\cos(\omega t + \varphi)\xi_0 - G_2\cos(\omega t + \varphi)\eta_0 + \cdots;
\end{aligned} \tag{49}$$

$$\frac{\partial^2\xi_1}{\partial T_0^2} + \xi_1 = -2\frac{\partial^2\xi_0}{\partial T_0\partial T_1} - \alpha_1\frac{\partial\xi_0}{\partial T_0} + 2\bar{\gamma}c(\beta\xi_0 + c\eta_0). \tag{50}$$

Summands, which do not give a contribution to the secular terms, are not written in the Equation (49). The following system of modulation equations is derived from (49, 50):

$$\begin{aligned}
&\dot{a}_1 + \frac{\alpha_1}{2}a_1 = 0; \; \dot{\theta}_1 + \bar{\gamma}c\beta = 0;\\
&\dot{a}_2 + \frac{\beta_1}{2}a_2 + \frac{\gamma_2}{2}a_2\sin(\delta_1T_1 + \varphi - 2\theta_2) = 0,\\
&-\dot{\theta}_2a_2 + \frac{\gamma_1}{2}a_2 + \frac{\gamma_2}{2}a_2\cos(\delta_1T_1 + \varphi - 2\theta_2) = 0.
\end{aligned} \tag{51}$$

The values γ_1 and γ_2 are given in Appendix 2. The first equation of the system (51) has only one fixed point $a_1 = 0$. The variables (x, y) are introduced to study the stability of the trivial solutions: $(x, y) = a_2[\cos(\frac{\delta_1}{2}T_1 + \frac{\varphi}{2} - \theta_2),\ \sin(\frac{\delta_1}{2}T_1 + \frac{\varphi}{2} - \theta_2)]$. Two modulation equations of the system (51) have the following form with respect to (x, y):

$$\begin{aligned}
\dot{x} &= -\frac{\alpha_2}{2}x - \frac{1}{2}(\delta_1 - \gamma_1 + \gamma_2)\,y;\\
\dot{y} &= -\frac{\alpha_2}{2}y + \frac{1}{2}(\delta_1 - \gamma_1 - \gamma_2)\,x.
\end{aligned} \tag{52}$$

The boundaries of stability/instability for the trivial solutions of the system (52) are analyzed now. The solutions are presented in the following form: $(x, y) = \exp(\lambda t)(X, Y)$. Then the boundary of instability can be obtained as

$$(\gamma_2 - \delta_1 + \gamma_1)(\gamma_2 + \delta_1 - \gamma_1) = \alpha_2^2. \tag{53}$$

To rewrite the system (53) in a more suitable form, the parameters γ_1 and γ_2 are presented as

$$\gamma_1 = \frac{2\sqrt{2}psc^2\bar{\gamma}}{2p^2s^2 - 1}; \quad \gamma_2 = \frac{\sqrt{2}cpf(1 - 2c^2)}{(1 - 8p^2s^2)s}. \tag{54}$$

Let us assume that the snap-through truss is shallow. Then the value s is small and the inequality $\gamma_1 \ll \gamma_2$ is satisfied. The boundary of the unstable domain has the following form:

$$a^2 f^2 = \delta_1^2 + \alpha_2^2, \tag{55}$$

where $a = \frac{\sqrt{2}cp}{s}$. Note, that the equations

$$\eta = x \cos\frac{\omega t + \varphi}{2} + y \sin\frac{\omega t + \varphi}{2} + O(\varepsilon); \quad \xi = 0 + O(\varepsilon) \tag{56}$$

connect the variables (x, y) and the variables (ξ, η) of the system (46).

The results of the numerical calculations of the boundary (55) are shown on Figure 3. These calculations are performed with the following numerical values of the system parameters: $p = 1$; $\varphi = 0.15$; $\alpha_2 = 1$. Two straight lines (Figure 3) are the boundaries of the instability regions of the undamped system. The boundaries of the instability regions of the damped system are shown on this figure too. The oscillations are unstable in the shaded region.

5.2. Resonance $\sqrt{2}ps \approx 2$, $\omega \approx 1$

Periodical solutions of the system (43) are considered at $k = 2$. It is assumed that the internal resonance 1:2 is met:

$$\sqrt{2}ps \approx 2 + \varepsilon\Delta_1. \tag{57}$$

Once more resonance condition $\omega = 1 + \varepsilon\sigma$ is taken into account. Using the multiple scales method, the next system of modulation equations is derived:

$$\begin{aligned}
\dot{a}_1 + \frac{\alpha_1}{2}a_1 - \frac{f}{2}\sin(\sigma T_1 + \varphi - \theta_1) &= 0;\\
a_1\dot{\theta}_1 + \bar{\gamma}c\beta a_1 + \frac{f}{2}\cos(\sigma T_1 + \varphi - \theta_1) &= 0,\\
2\dot{a}_2 + \alpha_2 a_2 + p^2\lambda\frac{a_1^2}{4}\sin(2\theta_1 - \theta_2 - \Delta_1 T_1) &= 0,\\
-2\dot{\theta}_2 a_2 + 4c\bar{\gamma}\beta a_2 + p^2\lambda\frac{a_1^2}{4}\cos(2\theta_1 - \theta_2 - \Delta_1 T_1) &= 0.
\end{aligned} \tag{58}$$

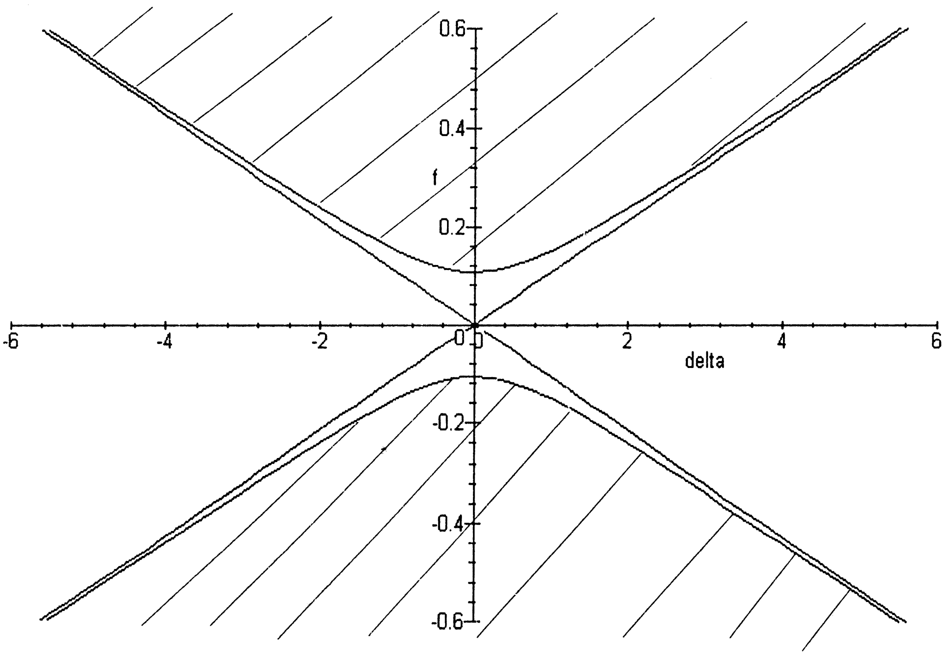

Figure 3. The boundaries of stable/unstable solutions in the parameter plane (δ_1, f). These boundaries correspond to the following parameters values: $p = 1$; $\varphi = 0.15; \alpha_2 = 1$. Two straight lines are the boundaries of the instability of the undamped system. The boundaries of the instability regions of the damped system are shown in this figure too. The oscillations are unstable in the shaded region.

System (58) is nonautonomous. It is transformed into the autonomous system due to the change of variables: $(\alpha_1, \alpha_2) = (\sigma T_1 + \varphi - \theta_1, 2\theta_1 - \theta_2 - \Delta T_1)$. The fixed points of this autonomous system are obtained from the equations: $\dot{a}_1 = \dot{a}_2 = \dot{\alpha}_1 = \dot{\alpha}_2 = 0$. Periodical solutions of the system (43) can be presented as

$$x = a_1 \cos(\omega t + \varphi - \alpha_1) + O(\varepsilon); \quad y = a_2 \cos(2\omega t + 2\varphi - 2\alpha_1 - \alpha_2) + O(\varepsilon), \tag{59}$$

where

$$a_1 = \frac{f}{\sqrt{\alpha_1^2 + 4(\sigma + \bar{\gamma} c\beta)^2}}; \quad a_2^2 = \frac{p^4 \lambda^2 f^4}{16\left[\alpha_1^2 + 4(\sigma + \bar{\gamma} c\beta)^2\right]^2 \left[\alpha_2^2 + (4\sigma - 2\Delta_1 - 4c\bar{\gamma}\beta)^2\right]}.$$

The numerical calculations of the frequency responses $a_1(\sigma)$ and $a_2^2(\sigma)$ are performed with the following numerical values of the system parameters: $\alpha_1 = 0.15$; $\alpha_2 = 0.02$; $\bar{\gamma} = 1$; $f = 0.1$; $\varphi = 0.36$; $p = 4$. Figure 4 shows the results of the numerical calculations. The frequency response $a_1(\sigma)$ is similar to one of a linear system. More interesting is the frequency response $a_2^2(\sigma)$. It has two peaks in the vicinity of one resonance. The similar response is presented in the book [17].

Let us study the stability of fixed points of system (58). The eigenvalues of the linearized flow $\lambda_1, \ldots, \lambda_4$ have the following form:

$$\lambda_{1,2} = -\frac{\beta_1}{2} \pm i\,|2\bar{\gamma} c\beta - 2\sigma + \Delta_1|; \quad \lambda_{3,4} = -\frac{\beta_2}{2} \pm i\,|\sigma + \bar{\gamma} c\beta|. \tag{60}$$

Thus, the periodic motions (Figure 4) are stable.

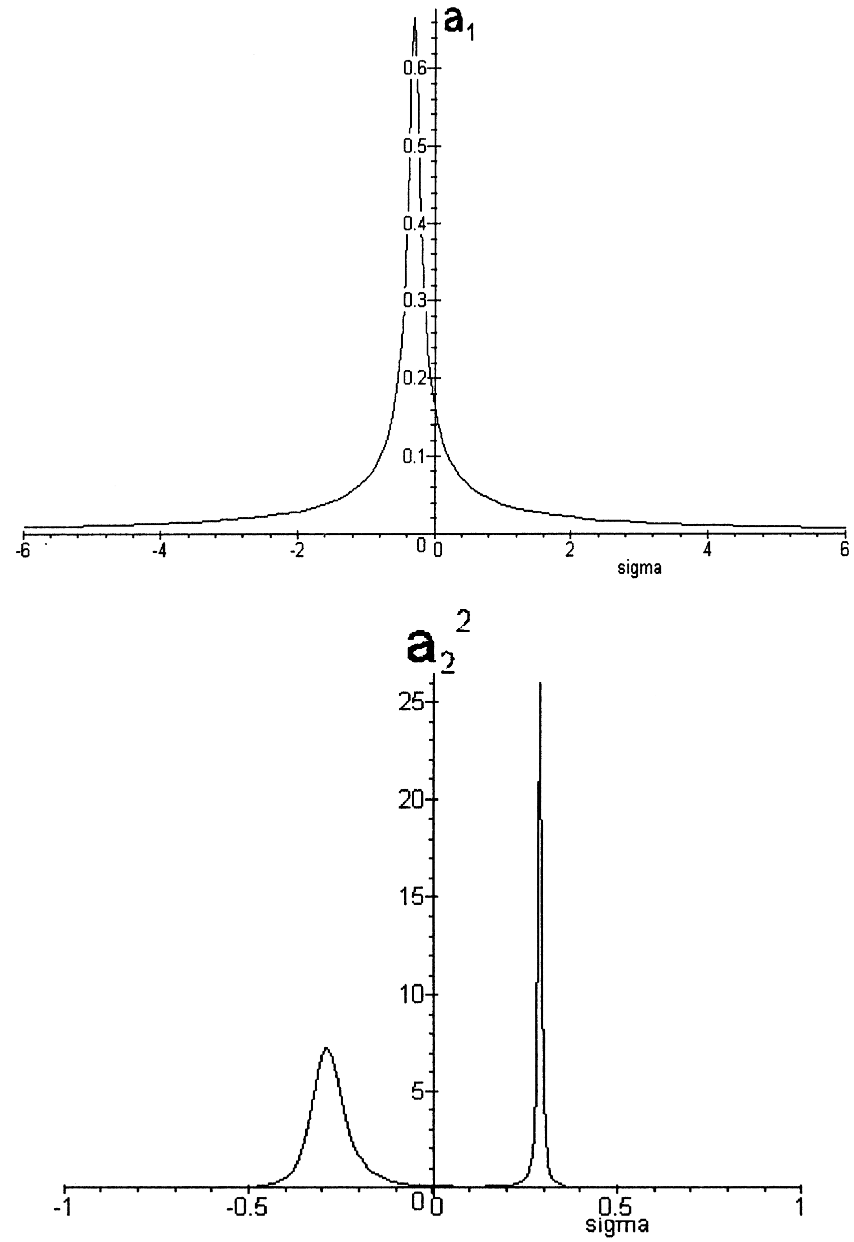

Figure 4. The frequency responses $a_1(\sigma)$ and $a_2^2(\sigma)$, which are calculated with the following numerical values of system parameters: $\alpha_1 = 0.15; \alpha_2 = 0.02; \bar{\gamma} = 1; f = 0.1; \varphi = 0.36; p = 4$.

5.3. Resonance with the Frequency Responses Similar to Linear Systems

Different resonances with the frequency responses, which are similar to linear systems, were discovered in Equations (43). These results, which are derived by the multiple scales method are presented in this section. The algebra leading to these results is omitted.

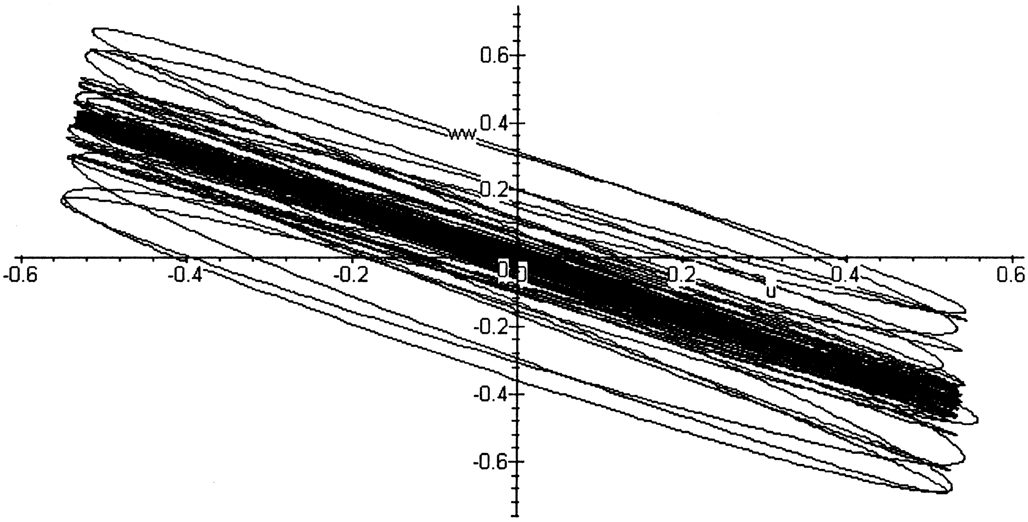

Figure 5. The results of the numerical simulations in the configuration plane.

1. Resonance $\omega = 1 + \varepsilon\,\sigma$.

$$x = \frac{f}{\sqrt{\alpha_1^2 + 4(\sigma + \bar{\gamma} c\beta)^2}} \cos(\omega t + \varphi - \alpha_1) + O(\varepsilon), \quad y = 0 + O(\varepsilon).$$

2. Resonance $\omega = \sqrt{2}sp + \varepsilon\Delta$

$$x = 0 + O(\varepsilon); \quad y = \frac{\alpha f}{sp\sqrt{2\alpha_2^2 + 8(\Delta - \sqrt{2}psc\bar{\gamma}\beta)^2}} \cos(\omega t + \varphi - \alpha_2) + O(\varepsilon).$$

3. Resonance $\sqrt{2}ps = 2 + \varepsilon\Delta_1, \omega = \sqrt{2}sp + \varepsilon\Delta$.

$$x = 0 + O(\varepsilon), \quad y = \frac{\alpha f}{2\sqrt{\alpha_2^2 + (2\Delta - 4c\bar{\gamma}\beta)^2}} \cos(\omega t + \varphi - \alpha_2) + O(\varepsilon).$$

4. Resonance $2\sqrt{2}ps = 1 + \varepsilon\delta, \omega = 1 + \varepsilon\sigma$

$$x = \frac{f}{\sqrt{\alpha_1^2 + 4(\sigma + \bar{\gamma} c\beta)^2}} \cos(\omega t + \varphi - \alpha_1) + O(\varepsilon), \quad y = 0 + O(\varepsilon).$$

5. Resonance $\omega = \sqrt{2}ps + \varepsilon\Delta, 2\sqrt{2}ps = 1 + \varepsilon\delta$

$$x = 0 + O(\varepsilon), \quad y = \frac{\alpha f}{\sqrt{\frac{\alpha_2^2}{4} + (\Delta - 0.5c\bar{\gamma}\beta)^2}} \cos(\omega t + \varphi - \alpha_2) + O(\varepsilon).$$

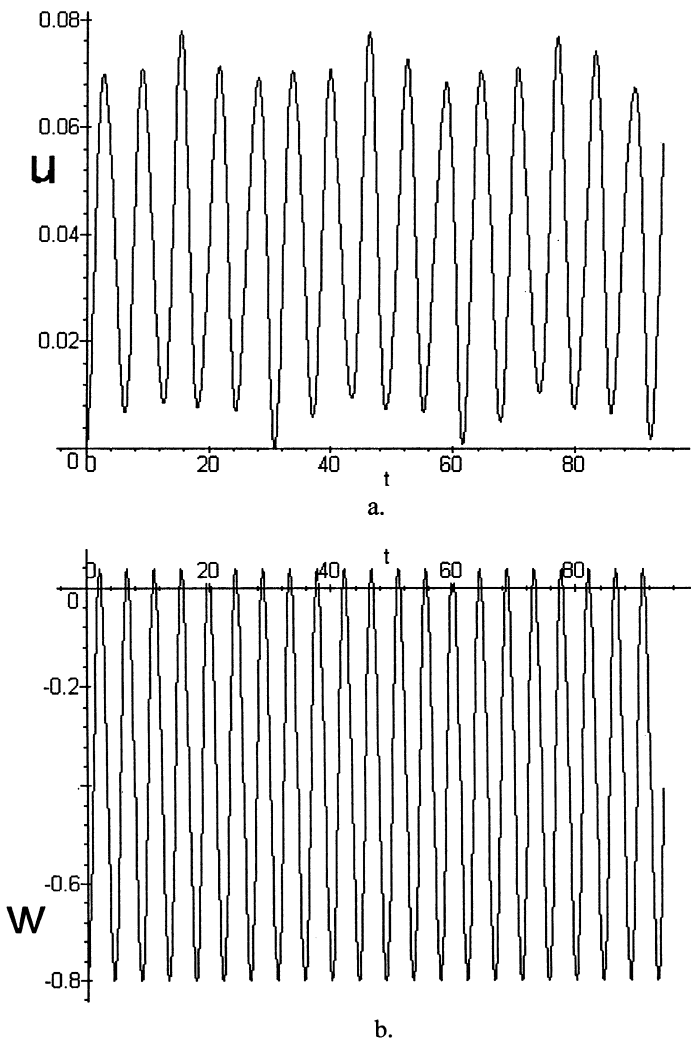

Figure 6. The dependences of the generalized coordinates u and w on t. Numerical calculations are performed with the following parameters: $\gamma = 0.05$; $\mu = 0.022$; $f = 1$; $\omega = 1.84$; $\varepsilon = 10^{-2}$; $s = 0.39$; $c = 0.92$; $p = 1.5$.

6. Numerical Simulations

Instability of the periodic oscillations, which is studied analytically in Section 5, is confirmed numerically. The system (45) is integrated with the following initial conditions:

$$u(0) = \frac{f}{1 - 8p^2s^2}; \quad w(0) = \alpha f\left(\frac{1}{1 - 8p^2s^2} + \frac{1}{6p^2s^2}\right); \quad \dot{u}(0) = \dot{w}(0) = 0.$$

The frequency ω is chosen from the resonance conditions (44) at $\delta_1 = 0$. The numerical values of the parameters are taken to be the following: $\bar{\gamma} = 1$; $\varepsilon = 0.01$; $f = 2.05$; $p = 2$; $c = 0.92$; $s = 0.39$; $\omega = 2.206$. The results of the numerical simulations are presented in the configuration plane. Note, that the numerically obtained trajectory is close to the straight line in the configuration plane (Figure 5). This trajectory is unstable, which confirm the analytical results.

To study the vibration absorption mode, the numerical calculations are performed for the differential Equations (4) with the following parameters: $\gamma = 0.05$; $\mu = 0.022$; $f = 1$; $\omega = 1.84$; $\varepsilon = 10^{-2}$; $s = 0.39$; $c = 0.92$; $p = 1.5$. The initial conditions are chosen in the form: $w(0) = -0.8$; $u(o) = \dot{u}(0) = \dot{w}(0) = 0$. Note that the value $w(0)$ corresponds to the equilibrium position of the snap-through truss. The dependences of the generalized coordinates u and w on t are shown in Figure 6. This figure shows the vibrations absorption mode, because the main mass has small amplitudes oscillations and the snap-through truss performs oscillations with the significant amplitudes.

7. Conclusion

The authors of this paper investigated the forced vibrations modes in the nonlinear system. We obtain that in this case the oscillations close to the stable equilibrium positions with small amplitudes of the snap-through truss and moderate amplitudes of the elastic system can be observed. These motions are not favorable for the elastic oscillations absorption. Therefore, the parameters of the snap- through truss must be chosen so that these motions are unstable. For examples, if the excitation frequency $\omega \approx 2\nu$ and the mechanical system parameters approximately satisfy the Equation (10), the parameters of the snap-through truss can be chosen so that they belong to the region of the unstable oscillations (Figure 2). If the mechanical system parameters approximately satisfy the Equation (10) and the excitation frequency $\omega \approx \nu_1$, the snap-through truss parameters can be chosen so that the resonance condition satisfies. This condition is derived from Equation (38). If the stiffness and the mass of the snap-through truss is significantly smaller than the corresponding parameters of the main system and $\omega \approx 2\sqrt{2}ps$, the system parameters can be chosen so that they belong to the region of the unstable oscillations (Figure 3). Moreover, authors analyzed forced motions when the snap-through truss jumps between two equilibria positions and the main system performs the oscillations with the small amplitudes [16].

Appendix A: The Parameters of the System (8)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \frac{\nu_2^2 - 2p^2s^2}{2p^2sc} \\ \frac{2p^2s^2 - \nu_2^2}{2\gamma cs} & 1 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix};$$

$$A_1 = \chi_1^2 \left[a_1 + a_2 \frac{\left(2p^2s^2 - \nu_2^2\right)^2}{4\gamma^2c^2s^2} - a_3 \frac{2p^2s^2 - \nu_2^2}{2\gamma cs} \right];$$

$$A_2 = \chi_1^2 \left[a_1 \frac{\left(2p^2s^2 - \nu_2^2\right)^2}{4p^4c^2s^2} + a_2 + a_3 \frac{2p^2s^2 - \nu_2^2}{2p^2sc} \right];$$

$$A_3 = \chi_1^2 \left[a_1 \frac{2p^2s^2 - \nu_2^2}{p^2cs} - a_2 \frac{2p^2s^2 - \nu_2^2}{\gamma cs} + a_3 - a_3 \frac{\left(2p^2s^2 - \nu_2^2\right)^2}{4p^2\gamma c^2s^2} \right];$$

$$B_1 = \chi_1^2 \left[b_1 + b_2 \frac{\left(2p^2s^2 - \nu_2^2\right)^2}{4\gamma^2c^2s^2} - b_3 \frac{2p^2s^2 - \nu_2^2}{2\gamma cs} \right];$$

$$B_2 = \chi_1^2 \left[b_1 \frac{\left(2p^2s^2 - \nu_2^2\right)^2}{4p^4c^2s^2} + b_2 + b_3 \frac{2p^2s^2 - \nu_2^2}{2p^2sc} \right];$$

$$B_3 = \chi_1^2 \left[b_1 \frac{2p^2s^2 - \nu_2^2}{p^2cs} - b_2 \frac{2p^2s^2 - \nu_2^2}{\gamma cs} + b_3 - b_3 \frac{\left(2p^2s^2 - \nu_2^2\right)^2}{4p^2\gamma c^2s^2} \right];$$

$$\chi_1 = \frac{4\gamma p^2 c^2 s^2}{\left(\nu_2^2 - 2p^2 s^2\right)^2 + 4\gamma p^2 c^2 s^2}; \quad b_1 = \frac{s}{2}\left(4p^2 - 3\nu_2^2\right);$$
$$b_2 = \frac{\nu_2^2(3s^2-1) - 4p^2 s^2}{2s}; \quad b_3 = \frac{(3c^2-1)\left(2p^2 s^2 - \nu_2^2\right) + 2p^2 c^2}{c};$$
$$a_1 = \frac{1}{2c}\left[2\gamma(3c^2 s^2 + 3c^2 - 1) + (3c^2-1)\left(1-\nu_1^2\right)\right];$$
$$a_2 = -0.5\left\lfloor 2\gamma c(2+3s^2) + \left(1-\nu_1^2\right)3c\right\rfloor;$$
$$a_3 = \frac{1}{s}\left[2\gamma(3c^2 s^2 + c^2) + 1 - \nu_1^2\right];$$
$$\delta_1 = \frac{\chi_1\left(2p^2 s^2 - \nu_2^2\right)(\alpha_2 - \alpha_1)}{2p^2 sc}; \quad \delta_2 = \chi_1\left[\alpha_1 + \alpha_2 \frac{\left(\nu_2^2 - 2p^2 s^2\right)^2}{4p^2\gamma c^2 s^2}\right];$$
$$\delta_3 = \chi_1\left(\alpha_1 \frac{\left(\nu_2^2 - 2p^2 s^2\right)^2}{4\gamma p^2 c^2 s^2} + \alpha_2\right); \quad \tilde{\alpha} = \frac{2p^2 s^2 - \nu_2^2}{2\gamma cs};$$
$$D_{\mathrm{A}}^{(1)} = A_1 + \tilde{\alpha}^2 A_2 + A_3\tilde{\alpha}; \quad D_{\mathrm{A}}^{(2)} = 2\Lambda A_1 + A_3\tilde{\alpha}\Lambda;$$
$$D_{\mathrm{A}}^{(3)} = 2A_2\tilde{\alpha}\Lambda + A_3\Lambda; \quad D_{\mathrm{B}}^{(1)} = B_1 + \tilde{\alpha}^2 B_2 + B_3\tilde{\alpha};$$
$$D_{\mathrm{B}}^{(2)} = 2\Lambda B_1 + B_3\tilde{\alpha}\Lambda; \quad D_{\mathrm{B}}^{(3)} = B_2 2\tilde{\alpha}\Lambda + B_3\Lambda;$$
$$F_{\mathrm{B}}^{(1)} = \frac{\Lambda^2}{2} D_{\mathrm{B}}^{(1)}; \quad F_{\mathrm{A}}^{(1)} = \frac{\Lambda^2}{2} D_{\mathrm{A}}^{(1)}; \quad r = \frac{D_{\mathrm{A}}^{(3)} D_{\mathrm{B}}^{(2)}}{4\nu_1^2};$$
$$l = \frac{D_{\mathrm{B}}^{(3)^2}}{16\nu_1^2}; \quad n = \frac{D_{\mathrm{A}}^{(2)^2}}{16\nu_1^2}.$$

Appendix B: The Parameters of the System (41)

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2p^2 sc}{2p^2 s^2 - 1} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\alpha = \frac{2p^2 sc}{2p^2 s^2 - 1}; \quad \beta = \frac{c}{2p^2 s^2 - 1}; \quad \Lambda = 2c(2 + 3\beta c);$$
$$\lambda = 2\alpha\beta(3p^2 s^4 + 1 + p^2 s^2 - 3s^2) - (3c^2 - 1)s;$$
$$G_1 = \lambda p^2 2A + \Lambda p^2 B; \quad G_2 = 6sc^2 p^2 B + \Lambda p^2 A;$$
$$\gamma_1 = 2\sqrt{2} psc\bar{\gamma}\beta; \quad \gamma_2 = \frac{G_2}{2\sqrt{2} sp}.$$

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