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Determination of the Chaos Onset in Mechanical Systems with Several Equilibrium Positions

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Abstract. Determination of the chaos onset in some mechanical systems with several equilibrium positions are analyzed. Namely, the snap-through truss and the oscillator with a nonlinear dissipation force, under the external periodical excitation, are considered. Two approaches are used for the chaos onset determination. First, Padé and quasi-Padé approximants are used to construct closed homoclinic trajectories for a case of small dissipation. Convergence condition used earlier in the theory of nonlinear normal vibration modes as well conditions at infinity make possible to evaluate initial amplitude values for the trajectories with admissible precision. Mutual instability of phase trajectories is used as criterion of chaotic behavior in nonlinear systems for a case of not small dissipation. The numerical realization of the Lyapunov stability definition gives us a possibility to observe a process of appearance and fast enlargement of the chaotic behavior regions if some selected parameters of the dynamical systems under consideration are changing.

Key words: Padé approximants, Homoclinic orbits, Mutual unstability.

1. Introduction

Homo- and hetero-clinic trajectories (HT) have been extensively studied in the literature [1, 2]. A formation of HT is considered as a criterion of the chaos onset in dynamical systems. In most cases authors of last and recent publications on the HT construction use the well-known Melnikov condition of the trajectory formation [3–6], which gives us a single equation for a determination of all unknown parameters of the system corresponding HT formation. As a result in the Melnikov condition, separatrix trajectories of the corresponding autonomous equations that is HT of zero approximation are utilized. A problem of effective analytic approximation of HT of non-autonomous system is not solved in general case up to now. Here a new approach for the HT construction in the nonlinear systems with phase space of dimension equal to two for a case of small dissipation is utilized. Padé approximants (PA) and quasi-Padé approximants (QPA) [7] are used for a representation both the HT in the dynamical system phase space and the corresponding time solution. Convergence condition used earlier in the theory of nonlinear normal vibration modes

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[8–10] as well conditions at infinity made possible to solve the boundary-value problem formulated for the HT and evaluate initial amplitude values with admissible precision for small value of dissipation.

Besides, for not small values of dissipation we suggest an approach to determine the onset of chaos based on some consequences of the classical Lyapunov stability definition for a case when initial variations are not arbitrary small and limited below. Mutual instability of phase trajectories is accepted as a criterion of chaotic behavior in dynamical systems. One compares trajectories that are initially very close. The numerical realization of the Lyapunov stability definition shows the mutual stability or instability of the trajectories. Calculations permit to observe a process of appearance and fast enlargement of the chaotic behavior regions if some selected parameters of the nonlinear system under consideration are changing.

Concrete results on the HT construction were obtained previously [11] for the nonautonomous Duffing equation, self-oscillating system with cubic nonlinearity, parametrically excited nonlinear pendulum system and other systems. In this paper the system which contains a single-DOF oscillator connected with an essential nonlinear absorber under periodical external force and the one-degree-of-freedom weakly forced (quasi-autonomous) oscillator with a nonlinear dissipation characteristic [12] are considered.

2. Convergence Condition

Let's assume that there are local expansions of solution obtained at small and large values of a parameter c:

$$y^{(0)} = \sum_{j=0}^{\infty} a_j c^j, \quad y^{(\infty)} = \sum_{j=0}^{\infty} b_j c^{-j}. \tag{1}$$

In order to join local expansions (1), fractional rational diagonal two-point Padé approximants (PA) [13] can be used. Let's consider

$$\mathrm{PA}_s = \frac{\sum_{j=0}^{s} \alpha_j c^j}{\sum_{j=0}^{s} \beta_j c^j} = \frac{\sum_{j=0}^{s} \alpha_j c^{j-s}}{\sum_{j=0}^{s} \beta_j c^{j-s}} \quad (s = 1, 2, 3, \ldots). \tag{2}$$

Comparing expressions (1) and (2) and retaining only the terms with an order of $c^r (-s \le r \le s)$ we will obtain a system of $2(s+1)$ linear algebraic equations for a determination of coefficients α_j, β_j. Since generally the determinant of the system Δ_s is not equal to zero, the system has a single trivial exact solution. But we hope to obtain the PA (2) having non-zero coefficients, which corresponds to the retaining terms in the equation (1).

Without loss of generality it can be assumed that in the Padé approximants PA_s of the form (2) the coefficient $\beta_0 = 1$ (if $\beta_0 \neq 0$).

In other case if $\beta_0 = 0$ then we obtain from algebraic equations for a determination of coefficients $\alpha_j, \beta_j \left(j = \overline{0, s}\right)$ that $\alpha_0 = 0$ too. Remaining non-zero coefficients gives us the PA of the $s - 1$ order, and the above arguments may be repeated.

Now, the system of algebraic equations for determination of α_j, β_j becomes overdetermined. All of the unknown coefficients can be determined from $(2s+1)$

equations while the "residual" of this approximate solution can be obtained by substitution of all the coefficients into the remaining equation.

THEOREM 1. The residual of the Padé approximant is linear proportional to the value of Δ_s that is non-zero coefficients and consequently exact PA will be obtained in the given approximation by c only in the case when $\Delta_s = 0$.

Proof. Let's consider the arbitrary linear homogeneous system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0; \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0; \\ \qquad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0. \end{cases} \tag{3}$$

Let's evaluate the determinant of the corresponding matrix:

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ni} A_{ni}, \tag{4}$$

where A_{ni} is the cofactor of the element a_{ni}. Let $x_n = 1$ and find $x_k (k = \overline{1, n-1})$ solving the first $n-1$ inhomogeneous equations using the Cramer method (we assume now that the denominator of this system is not equal to zero):

$$x_k = -\frac{\Delta_k^{n-1}}{\Delta^{n-1}} (k = \overline{1, n-1}), \tag{5}$$

where

$$\Delta^{n-1} = \begin{vmatrix} a_{11} & \dots & a_{1n-1} \\ \vdots & \ddots & \vdots \\ a_{n-11} & \dots & a_{n-1n-1} \end{vmatrix},$$

and Δ_k^{n-1} is obtained from Δ^{n-1} by replacing the k-th column on the column $(a_{1n}, \dots, a_{n-1n})^T$, where T denotes the transposing. Let's find A_{ni} in this table of symbols:

$$A_{n1} = (-1)^{n+1} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-12} & \cdots & a_{n-1n} \end{vmatrix} = -\Delta_1^{n-1};$$

$$\cdots$$

$$A_{nk} = (-1)^{n+k} \begin{vmatrix} a_{11} & \cdots & a_{1k-1} & a_{1k+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-11} & \cdots & a_{n-1k-1} & a_{n-1k+1} & \cdots & a_{n-1n} \end{vmatrix} = -\Delta_k^{n-1};$$

$$\cdots$$

$$A_{n\ n-1} = (-1)^{n+n-1} \begin{vmatrix} a_{11} & \cdots & a_{1n-2} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-11} & \cdots & a_{1n-2} & a_{n-1n} \end{vmatrix} = -\Delta_{n-1}^{n-1};$$

$$A_{nn} = (-1)^{n+n} \begin{vmatrix} a_{11} & \dots & a_{1n-1} \\ \vdots & \ddots & \vdots \\ a_{n-11} & \dots & a_{n-1n-1} \end{vmatrix} = \Delta^{n-1}.$$

After replacing of these expressions in (4) and using (5) one obtains:

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ni} A_{ni} = -\sum_{i=1}^{n-1} a_{ni} \Delta_i^{n-1} + a_{nn} \Delta^{n-1}$$

$$= \Delta^{n-1} \left(\sum_{i=1}^{n-1} a_{ni} \frac{-\Delta_i^{n-1}}{\Delta^{n-1}} + a_{nn} \right) = \Delta^{n-1} \left(\sum_{i=1}^{n-1} a_{ni} x_i + a_{nn} \right).$$

If the determinant Δ^{n-1} is equal to zero then we can choose another equation as the last one. If in every choice of the extra equation the Δ^{n-1} is equals to zero then there are at least two linearly dependent equations what means that the determinant corresponding to $PA_{s-1} = \frac{\sum_{j=0}^{s-1} \alpha_j c^j}{\sum_{j=0}^{s-1} \beta_j c^j}$ is equal to zero and we may consider PA_{s-1}. □

Hence the following is a necessary condition for convergence of the succession of PA_s (2) at $s \to \infty$ to fractional rational function P_∞. Namely,

$$\lim_{s \to \infty} \Delta_s = 0. \tag{6}$$

It's possible to generalize condition (6) to quasi-Padé approximants (QPA) which contain powers of some unknown parameter and exponential functions. Besides, it is possible to utilize the condition (6) for obtaining some unknown parameters which are contained in local expansions [11].

3. Criterion of the Mutual Instability of the Phase Trajectories

Consider the well-known Lyapunov definition of stability stating that: the solution $y = 0$ is stable if for all positive ε there a positive σ exists such that for all $y_0 \in N_\sigma^{(0)}$ and $t \geq 0$ we have $y(t, y_0) \in N_\varepsilon^{(0)}$. Here $N_\alpha^{(0)} = \{y: \|y\| < \alpha\}$ where $\|.\|$ is some norm of the space.

Introduce a relation between the quantity ε and the initial value of the variable y. Let

$$\varepsilon = \rho \, \|y_0\| \leq \rho\sigma \, (\rho = \text{const}) \tag{7}$$

The condition (7) means that a value of δ is not arbitrarily small because $\sigma \geq \varepsilon/\rho$. One has from (7) that $\rho \geq \varepsilon/\sigma$, that is the constant ρ is a high limit of the fraction ε/σ. Besides, one obtains from the Lyapunov stability definition taking into account the inequality (7), that $\|y(t)\| \leq \rho \, \|y_0\|$.

Introducing the time of calculation T, one has from the preceding the following criterion:

Instability of the solution $y = 0$ is established if

$$\max_{0 \leq t \leq T} \|y(t)\| \geq \rho \, \|y(0)\| \quad (\rho > 0). \tag{8}$$

The proposed stability criterion (8) was obtained when the value of σ is not arbitrarily small and limited below. Note that this assumption does not contradict to the Lyapunov definition meaning because in this definition the initial values can not tend to zero.

It is necessary to choose values of ρ and T. Here a value ρ^{-1} is a measure of smallness of initial variations with respect to maximum admissible variations for any $t \geq 0$. An increase of the ρ means that feasible initial values of variation decrease. There is some arbitrariness in a choice of the value ρ. Really, in the instability region the variations leave the solution ε-neighborhood if t increases for any ρ. It should be taken into account that possible values of ρ in the region of stability can be small, so It should be chosen not small. Concrete calculations show that the choice of the value ρ equal to 10 permits to detect the fast enlargement of the instability regions. We define this situation as a passage from regular to chaotic behavior in nonlinear system.

Let's discusses now a choice of the constant T. Note that all concrete calculations are made at points on some chosen mesh of the system parameter space. Calculations are conducted as long as boundaries of stability/instability regions in a chosen scale on the system parameter space are varying. This is a criterion for the choice of the parameter T. It is clear that if the mesh widths decrease and the number of mesh points increases infinitely, the interval of time T tends to infinity.

We now discuss the dependence of the stability analysis on the variations initial conditions. The linear stability results are not dependent on initial conditions. But it is known [14, 15] that additional nonlinear instability regions (obtained if we take into account nonlinear terms) have a smaller dimension in parameter space than instability regions obtained by the linearized stability analysis. Numerical calculations verify that the stability analysis is independent of initial variations if the initial variations are small.

Remark. Note that in [16] some criterion is used which is similar to the criterion (3). But a choice of the calculation time T is not discussed in the work [16].

One introduces some mesh in the phase space region using the increments: $\Delta y, \Delta y'$. Points of the mesh $P_{ij}(y_{i0}, y'_{j0})$ will be chosen as initial points for the selected phase trajectories $y_{ij}^{(1)}(t)$. Let us take other initial points near the chosen initial points P_{ij}, namely $Q_{ij}(y_{i0} + \Delta y_0, y'_{j0})$ where the value Δy_0 is sufficiently small, and consider the other phase trajectory, $y_{ij}^{(2)}(t)$. Comparing the trajectories outgoing from the close initial points and using the criterion (8) we obtain the following.

Instability of the outgoing trajectory is established if

$$\left\| y_{ij}^{(1)}(t) - y_{ij}^{(2)}(t) \right\| \geq \rho \, \|\Delta y_0\| \quad (0 \leq t \leq T). \tag{9}$$

Note that a realization of the approach is not difficult because only the standard RK method with an additional stability condition is used.

4. The Snup-through Truss Motion

A snap-through truss is suggested to use for longitudinal oscillations absorption of an elastic solid [17]. In this case a part of an elastic oscillations energy is transferred to the truss, which is jumping from one equilibrium position to another. An elastic system is approximated by the single-DOF mass-spring model to study the truss capacity to absorb oscillations. By assumption, the truss is shallow and its mass and stiffness are significantly smaller then the corresponding parameters of the main elastic system. Such choice of the parameters is determined by the real absorber design conditions.

Figure 1 shows the system under consideration. The equations of motion without the external excitation are the following:

$$M\ddot{U}+\kappa_1 U+\kappa\left[U-L\cos\varphi+L\left\{1+\frac{W^2}{(L\cos\varphi-U)^2}\right\}^{-1/2}\right]=0;$$
$$m\ddot{W}+\kappa W\left[2-\frac{L}{\sqrt{(L\cos\varphi-U)^2+W^2}}-\frac{L}{\sqrt{L^2\cos^2\varphi+W^2}}\right]=0, \tag{10}$$

where (U, W) are the generalized coordinates; L is a length of the spring; φ is the angle, which defines the equilibrium position; κ is the truss spring stiffness; κ_1 is a stiffness of the main elastic system. The system (10) has three equilibrium positions: one saddle, $(U, W) = (\kappa L(\cos\varphi - 1)/(\kappa_1 + \kappa);\ 0)$ and two centers $(U, W) = (0; \pm L\sin\varphi)$.

Let the dimensionless variables $u = U/L;\ \ w = W/L$ and dimensionless time $t = \sqrt{M/\kappa_1}\tau$ be introduced. One introduces the variable $u_1 = u + \frac{\gamma(1-\kappa)}{1+\gamma}$, too. In the new variables the origin corresponds to the saddle. By assumption, the mass and stiffness of the truss are significantly smaller than the corresponding parameters of the linear sub-system. Therefore, the following relations are introduced: $\mu = \varepsilon\bar{\mu}, \gamma = \varepsilon\bar{\gamma}, \varepsilon \ll 1$. It is supposed that the snap-truss system is shallow. Retaining linear, quadratic and cubic terms by u_1, w we can rewrite the system (10) as

$$\ddot{u}_1+(1+\varepsilon\bar{\gamma})u_1-\tfrac{\varepsilon\bar{\gamma}}{\rho^3}u_1w^2-\tfrac{\varepsilon\bar{\gamma}}{2\rho^2}w^2=0;$$
$$\bar{\mu}\ddot{w}-\bar{\gamma}\alpha^2 w-\tfrac{\bar{\gamma}}{\rho^2}wu_1+\tfrac{\bar{\gamma}\beta^2}{2}w^3=0, \tag{11}$$

where

$$\rho=\frac{\gamma+\kappa}{1+\gamma};\ \alpha^2=\frac{1}{\rho}+\frac{1}{\kappa}-2;\ \beta^2=\frac{1}{\rho^3}+\frac{1}{\kappa^3}.$$

Under the external harmonic force a solution of the first equation of the system (11) can be presented in the zero approximation by ε of the form $u = F\cos(\omega t)$. Using the additional transformation (expansion) of variables and including the dissipation term we obtain from the second equation of the system (11) the next equation

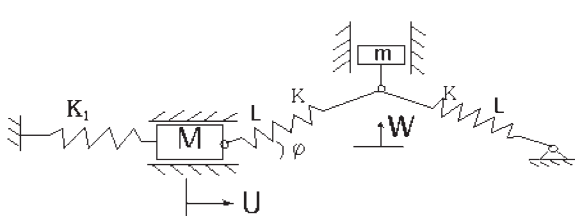

Figure 1. A scheme of a system containing the snap-through truss.

for the investigation:

$$y'' + \delta y' - (1 + f\cos(\omega t))\, y + y^3 = 0. \tag{12}$$

For the construction of HT we need in the information about the initial point (a_0, a_1) corresponding to $t = 0$ and about a dependence between the system parameters, namely the frequency ω, the amplitude f and the dissipation δ. Thus we should construct the algebraic system for a determination of the unknown values.

Consider the equation in the next form:

$$y'' + \delta y' - (1 + f\cos(\omega t + \varphi))\, y + y^3 = 0. \tag{13}$$

The phase φ allows to choose point $(a_0, 0)$ as initial one. Assume that $(y, y') \underset{t\to\pm\infty}{\longrightarrow} (0, 0)$ and that the sought solution is analytical one. Then we can consider the Taylor expansion of the solution $y(t)$:

$$y = a_0 + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots, \tag{14}$$

where a_0 is an arbitrary constant and $a_j = a_j\,(a_0, \varphi, f, \delta)\left(j = \overline{2, \infty}\right)$.

Multiplying the equation (13) by $y'(t)$ and integrating within the limits from $t = 0$ to $t = +\infty$ and from $t = 0$ to $t = -\infty$ we obtain the equations:

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} + \int_0^{+\infty} \left(\delta y' - yf\cos(\omega t + \varphi)\right) y' \mathrm{d}t = 0; \tag{15}$$

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} + \int_0^{-\infty} \left(\delta y' - yf\cos(\omega t + \varphi)\right) y' \mathrm{d}t = 0. \tag{16}$$

Linearizing the equations by δ we evaluate the integrals from (15)–(16) along the separatrix of the autonomous equation $y_{\mathrm{aut}} = \sqrt{2}/\cosh(t)$:

$$\begin{aligned}&\int_0^{\pm\infty} \left(\delta y'_{\mathrm{aut}} - y_{\mathrm{aut}} f\cos(\omega t + \varphi)\right) y'_{\mathrm{aut}} \mathrm{d}t \\ &\quad = \delta \int_0^{\pm\infty} y'^2_{\mathrm{aut}} \mathrm{d}t - \cos\varphi \int_0^{\pm\infty} y_{\mathrm{aut}} f\cos(\omega t) y'_{\mathrm{aut}} \mathrm{d}t + \sin\varphi \int_0^{\pm\infty} y_{\mathrm{aut}} f\cos(\omega t) y'_{\mathrm{aut}} \mathrm{d}t,\end{aligned}$$

where

$$\int_0^{+\infty} y'^2_{\mathrm{aut}} \mathrm{d}t = \int_{-\infty}^{0} y'^2_{\mathrm{aut}} \mathrm{d}t = \frac{2}{3};$$

$$\begin{aligned}\int_0^{+\infty} \sin(\omega t) y_{\mathrm{aut}} y'_{\mathrm{aut}} \mathrm{d}t &= \int_{-\infty}^{0} \sin(\omega t) y_{\mathrm{aut}} y'_{\mathrm{aut}} \mathrm{d}t = \int_0^{+\infty} \sin(\omega t) \frac{-2\sinh t}{\cosh^3 t} \mathrm{d}t \\ &= -\omega \int_0^{+\infty} \frac{\cos(\omega t)}{\cosh^2 t} \mathrm{d}t = -\frac{\omega^2 \pi}{2\sinh \frac{\omega\pi}{2}};\end{aligned}$$

$$\int_0^{+\infty}\cos(\omega t)y_{\text{aut}}y'_{\text{aut}}\mathrm{d}t=-\int_{-\infty}^{0}\cos(\omega t)y_{\text{aut}}y'_{\text{aut}}\mathrm{d}t$$
$$=\int_0^{+\infty}\cos(\omega t)\frac{-2\sinh t}{\cosh^3 t}\mathrm{d}t=\omega\int_0^{+\infty}\frac{\sin(\omega t)}{\cosh^2 t}\mathrm{d}t.$$

Substituting these expressions into equations (15)–(16) we obtain:

$$\frac{a_0^2}{2}-\frac{a_0^4}{4}-\frac{2}{3}\delta-\frac{\omega^2\pi f\sin\varphi}{2\sinh\frac{\omega\pi}{2}}-f\omega\cos\varphi\int_0^{+\infty}\frac{\sin(\omega t)}{\cosh^2 t}\mathrm{d}t=0; \tag{17}$$

$$\frac{a_0^2}{2}-\frac{a_0^4}{4}+\frac{2}{3}\delta+\frac{\omega^2\pi f\sin\varphi}{2\sinh\frac{\omega\pi}{2}}-f\omega\cos\varphi\int_0^{+\infty}\frac{\sin(\omega t)}{\cosh^2 t}\mathrm{d}t=0, \tag{18}$$

where the integral may be evaluated numerically.

Remark. The well-known Melnikov condition of the form

$$\delta\frac{2}{3}+\frac{\omega^2\pi f\sin\varphi}{2\sinh\frac{\omega\pi}{2}}=0 \tag{19}$$

can be obtained from the equations (17) and (18). Note the condition is not used here to solve the problem of homoclinic trajectory construction.

For the continuation the local expansion ad infinitum we rebuild it to QPA of the form

$$y=a_0+a_2t^2+a_3t^3+a_4t^4+\cdots\to e^{-t}\frac{\alpha_0+\alpha_1e^t+\alpha_2e^{2t}}{1+\beta_1e^t+\beta_2e^{2t}}. \tag{20}$$

So, the additional equation may be obtained using the convergence condition (6). Note, that a derivation of the next equation is presented in the Appendix. One has the following:

$$-24a_0a_4a_2+144a_4^2a_0+144a_5a_2^2-288a_3a_4a_2+a_0a_2^2+144a_2a_3^2$$
$$+60a_3a_2^2-144a_4a_2^2+144a_3^3+12a_2^3-144a_5a_3a_0=0. \tag{21}$$

Nonlinear algebraic equations (17), (18) and (21) form the system for determination unknown parameters a_0,φ and $f=f(\omega)$ while the dissipation coefficient δ is fixed.

Figure 2 shows the dependences between the parameters of the system corresponding to HT and obtained from the proposed here method. Also the example of homoclinic trajectory and comparison of the trajectory evaluated by using RK method and using QPA are presented.

This approach is useful for the system investigation when the dissipation δ is small. For larger values of this parameter the criterion of mutual instability of the phase trajectories is used for investigation.

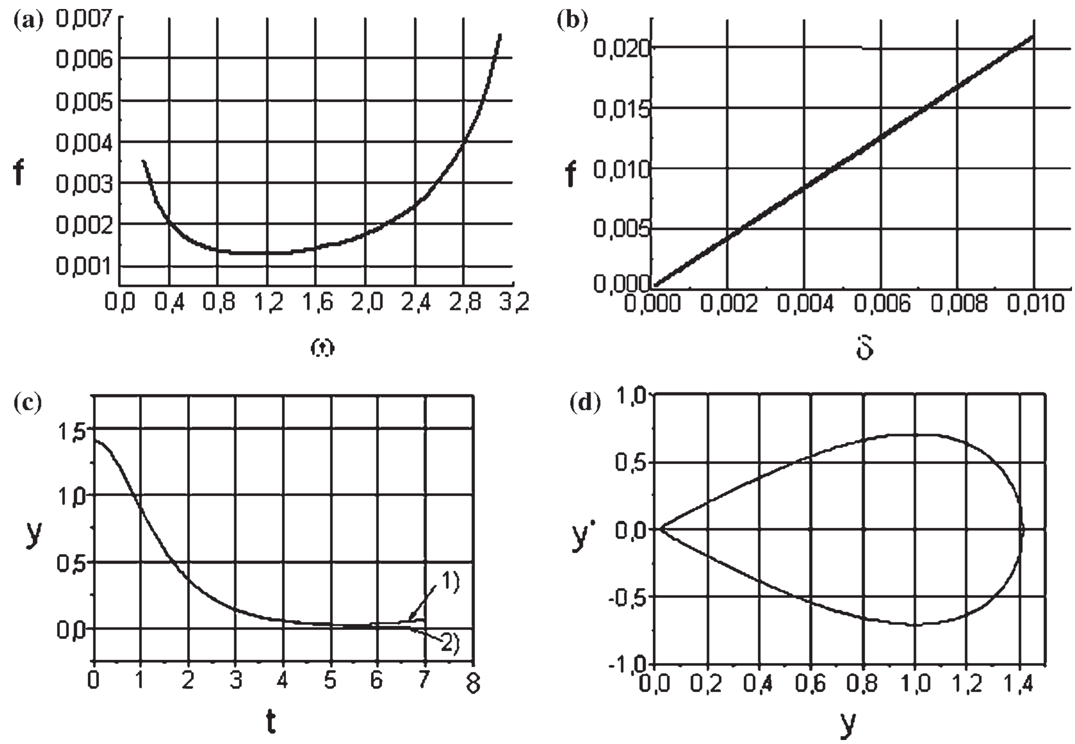

Figure 2. (a)–(b): Lower chaotic behavior boundaries in parameter spaces; (c): the comparison of the HT obtained from the RK method (curve 1) and by using QPA (22) (curve 2) while $\omega = 1, \delta = 0.001$; (d) homoclinic trajectory in phase space while $\omega = 1, \delta = 0.001$.

Let's examine the following region of the equation (13) phase plane: $0 \le y \le 1.6$, $0 \le y' \le 0.8$. Introduce some mesh in the defined region using the increments: $\Delta y = 0.02$, $\Delta y' = 0.016$. We utilize the criterion (11), where the value $\Delta y_0 = 0.1\Delta y$. It is selected $\rho = 10$. Results of the mutual instability analysis (the time of stabilization here $T = 50$) are presented in Figure 3 for different values of the external amplitude and the dissipation coefficient. Here the initial points of the chosen mesh, which correspond to unstable trajectories, are marked by dark squares. The calculations (in the chosen mesh of the equation phase place) show that for small value of f the mutual instability of phase trajectories can be observed near the separatrix branches. Instability regions begin to extend if values of the external amplitude f are increasing, and this enlargement is very fast.

5. The One-degree-of-freedom Weakly Forced Oscillator with Nonlinear Dissipation Forces

Mechanical system with a small periodic external excitation, nonlinear dissipation forces and the Duffing type stiffness is governed by the following second order differential equation (12):

$$y'' - y + y^3 = f\cos(\omega t + \varphi) - \theta\left(y' - \nu^*\right), \tag{22}$$

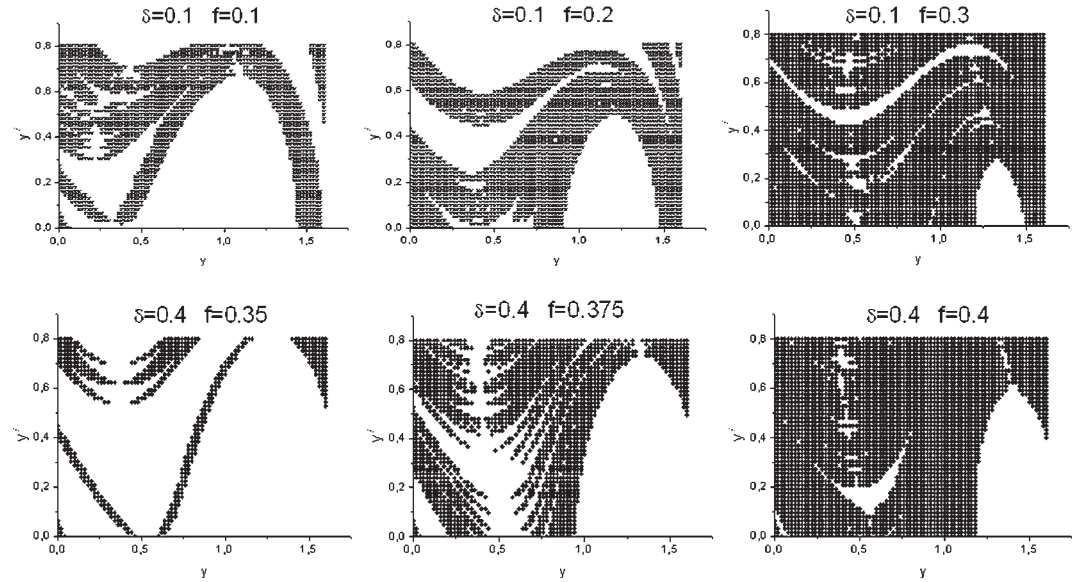

Figure 3. Mutual stability/instability of phase trajectories of the snup-through truss motion equation (13). Results of the stability analysis are obtained for $\omega = 1, T = 50$. The initial points of the chosen mesh which correspond to unstable trajectories are marked by dark squares.

where $\theta\left(y' - \nu^*\right) = T_0 \operatorname{sign}\left(y' - \nu^*\right) - \alpha\left(y' - \nu^*\right) + \beta\left(y' - \nu^*\right)^3$ is the nonlinear dissipation characteristic.

For a construction of homoclinic trajectory we need to know about the initial point $(a_0, 0)$ and the phase φ corresponding to $t = 0$ and the system parameters, namely ω, f and θ. Thus we should construct the algebraic system for determination of unknown values.

Let's make some assumption like for the previous system. One assume that $\left(y, y'\right) \underset{t\to\pm\infty}{\longrightarrow} (0, 0)$. We will construct the analytical approximation for the sought solution. First, we can consider the Taylor expansion at zero of the solution $y(t)$:

$$y = a_0 + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots, \tag{23}$$

where a_0 is an arbitrary constant, $a_j = a_j\left(a_0, \varphi, f, T_0, \alpha, \beta\right)\left(j = \overline{2, \infty}\right)$.

Then multiplying the equation (22) by $y'(t)$ and integrating within the limits from $t = 0$ to $t = +\infty$ and from $t = 0$ to $t = -\infty$ we obtain the following equations where several integrals are calculated along the zero approximation $y_0 = \sqrt{2}/\cosh(t)$:

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} - \left(\alpha\nu^* - \beta\nu^{*3} - T_0\right) a_0 - \frac{2\alpha}{3} + \frac{8\beta}{35} + \frac{4\sqrt{2}\beta\nu^*}{5} + 2\beta\nu^{*2} +$$
$$+ f\sin\varphi \int_0^{+\infty} \sin(\omega t) y_0' \,\mathrm{d}t - f\cos\varphi \int_0^{+\infty} \cos(\omega t) y_0' \mathrm{d}t = 0; \tag{24}$$

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} - \left(\alpha\nu^* - \beta\nu^{*3}\right)a_0 + \frac{2\alpha}{3} - \frac{8\beta}{35} + \frac{4\sqrt{2}\beta\nu^*}{5} - 2\beta\nu^{*2}$$
$$- f\sin\varphi\int_0^{+\infty}\sin(\omega t)y_0'\,\mathrm{d}t - f\cos\varphi\int_0^{+\infty}\cos(\omega t)y_0'\,\mathrm{d}t$$
$$- T_0\int_{-\infty}^{0}\operatorname{sign}\left(y_0' - \nu^*\right)\,y_0'\,\mathrm{d}t = 0. \tag{25}$$

There

$$\int_0^{+\infty}\sin(\omega t)y_0'\mathrm{d}t = \int_{-\infty}^{0}\sin(\omega t)y_0'\mathrm{d}t = -\frac{\omega\sqrt{2}\pi}{2}\cdot\frac{1}{\cosh\frac{\omega\pi}{2}};$$

$$\int_0^{+\infty}\cos(\omega t)y_0'\mathrm{d}t = -\int_{-\infty}^{0}\cos(\omega t)y_0'\mathrm{d}t$$

$$= -\sqrt{2} + \omega\sqrt{2}\left(-\frac{\pi}{2}\tanh\frac{\omega\pi}{2} + 4\omega\sum_{k=0}^{\infty}\frac{1}{\omega^2 + (1+4k)^2}\right).$$

The integral $\int_{-\infty}^{0}\operatorname{sign}\left(y_0' - \nu^*\right)y_0'\mathrm{d}t$ is evaluated as a function of the parameter ν^* computationally.

For the continuation the local expansion ad infinitum we rebuild it to QPA:

$$y = a_0 + a_2t^2 + a_3t^3 + a_4t^4 + \cdots \to e^{-t}\frac{\alpha_0 + \alpha_1 e^t + \alpha_2 e^{2t}}{1 + \beta_1 e^t + \beta_2 e^{2t}}. \tag{26}$$

So, the additional equation may be obtained using the convergence equation (6). It is the same to the equation (21).

Nonlinear algebraic equations (24), (25) and (21) form the system for determination unknown parameters a_0, φ and $f = f(\omega)$ while the dissipation parameters T_0, α, β are fixed.

Figure 4 shows the dependences between the parameters of the system corresponding to HT and obtained from the proposed here method. Also the example of homoclinic trajectory and comparison of the trajectory evaluated by using the RK method and using QPA are presented.

For large values of this parameter we use the criterion of the mutual instability of the phase trajectories.

Let's examine the following region of phase plane for the equation (22): $0 \le y \le 1.6$, $0 \le y' \le 0.8$. One introduces some mesh in the defined region using the increments: $\Delta y = 0.02$, $\Delta y' = 0.016$. Let $\Delta y_0 = 0.1\Delta y$, $\rho = 10$.

Results of the mutual instability analysis (10) (the time of stabilization here $T = 50$) are presented in Figure 5 for different values of the external amplitude and the

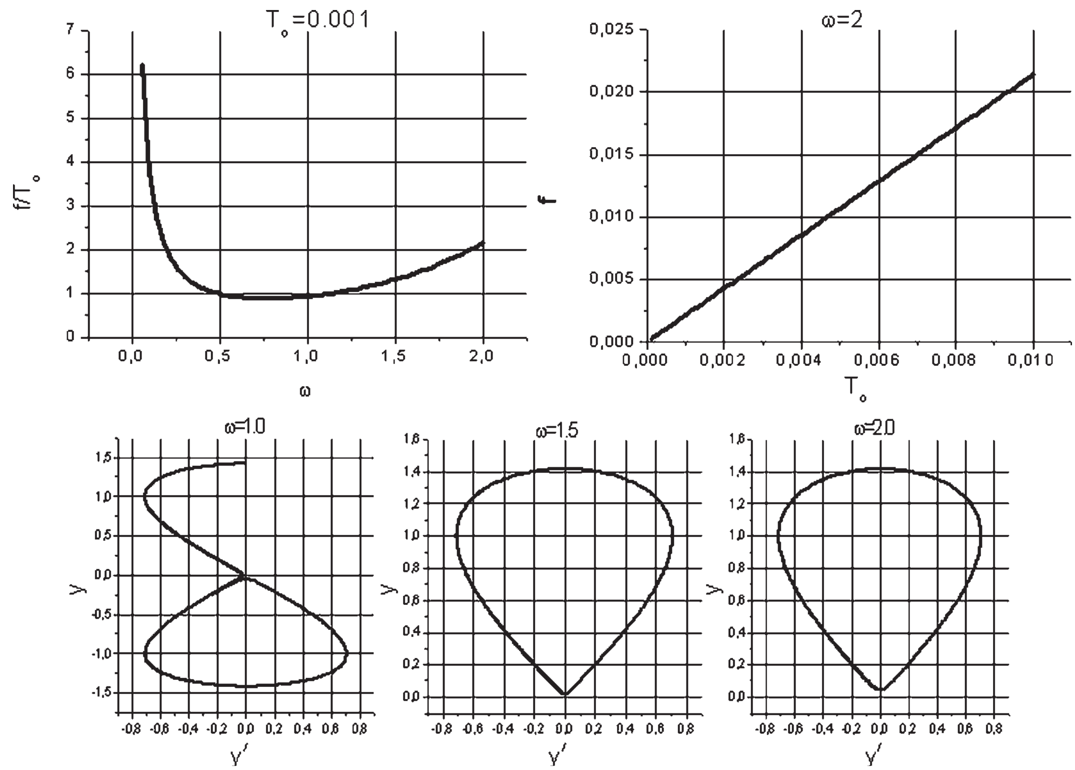

Figure 4. Chaotic behavior boundaries in parameter spaces and the homoclinic trajectories in phase space while $\nu = 0.5, T_0 = \alpha = \beta = 0.001$.

dissipation. Here the initial points of the chosen mesh, which correspond to unstable trajectories, are marked by dark squares. We can see the fast enlargement of the instability regions when values of the external amplitude f are increasing.

6. Conclusions

We considered here a determination of the chaos onset in some mechanical systems. One approach permits to construct closed homo- and hetero-clinic trajectories for a case of small dissipation. This approach is based on utilization as Padé and/or quasi-Padé approximants for a representation of the trajectories. Besides, it is possible to investigate the mutual instability of phase trajectories in a region of chaotic behavior fore a case of large dissipation by using some conclusions from the classical Lyapunov stability definition. Realization of the approaches in concrete mechanical systems gives us a possibility to state that the approaches presented here are sufficiently general to be applicable to other types of nonlinear systems of phase dimension equal to two or three. Unfortunately, we can not indicate for now how to determine a boundary of validity of the approaches in a space of parameters in systems under consideration. This could be a subject of some future investigation.

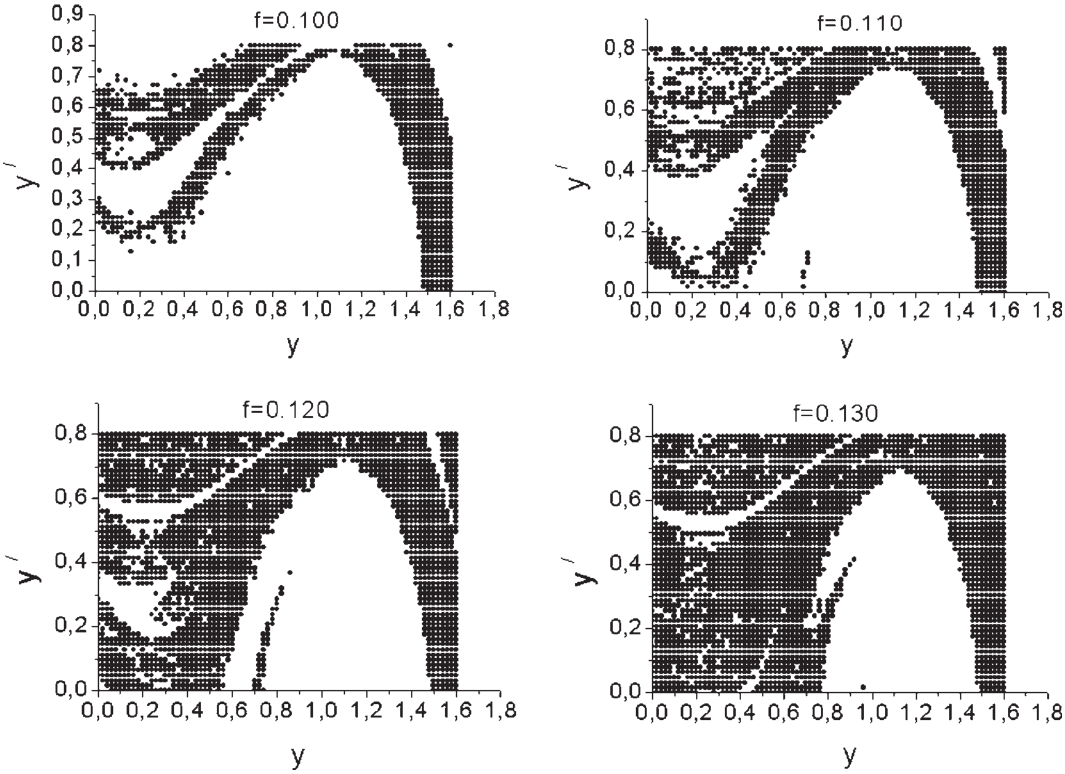

Figure 5. Mutual stability/instability of phase trajectories of the (22). Results of the stability analysis are obtained for $\omega=1, T_0=\alpha=\beta=0.1, T=50$. The initial points of the chosen mesh which correspond to unstable trajectories are marked by dark squares.

Appendix

From the equation (20) we have

$$a_0+a_2t^2+a_3t^3+a_4t^4+\cdots\approx e^{-t}\frac{\alpha_0+\alpha_1e^t+\alpha_2e^{2t}}{1+\beta_1e^t+\beta_2e^{2t}}. \qquad \text{(A.1)}$$

Let's rewrite (A.1) multiplying the local expansion by denominator of Padé approximant:

$$(a_0+a_2t^2+a_3t^3+\cdots)(1+\beta_1e^t+\beta_2e^{2t})\approx e^{-t}(\alpha_0+\alpha_1e^t+\alpha_2e^{2t}). \qquad \text{(A.2)}$$

Expanding the right and the left sides of (A.2) in Taylor series at $t=0$ and retaining only terms with an order of $t^r(0\le r\le 5)$ we obtain the system of 6 linear algebraic equations for calculating the coefficients of the considered Padé approximant:

$$a_0(1+\beta_1+\beta_2)-\alpha_0-\alpha_1-\alpha_2=0;$$
$$-\alpha_2+\alpha_0+a_0(2\beta_2+\beta_1)=0;$$
$$-1/2\alpha_2-1/2\alpha_0+a_0(2\beta_2+1/2\beta_1)+a_2(1+\beta_1+\beta_2)=0;$$
$$-1/6\alpha_2+1/6\alpha_0+a_0(4/3\beta_2+1/6\beta_1)+a_3(1+\beta_1+\beta_2)+a_2(2\beta_2+\beta_1)=0;$$

$$a_2(2\beta_2+1/2\beta_1)+a_0(1/24\beta_1+2/3\beta_2)+a_3(2\beta_2+\beta_1)+a_4(1+\beta_1+\beta_2)$$
$$-1/24\alpha_2-1/24\alpha_0=0;$$
$$-1/120\alpha_2+1/120\alpha_0+a_4(2\beta_2+\beta_1)+a_2(4/3\beta_2+1/6\beta_1)+a_0(4/15\beta_2+1/120\beta_1)$$
$$+a_5(1+\beta_1+\beta_2)+a_3(2\beta_2+1/2\beta_1)=0.$$

Then from the first five equations we have

$$\beta_1=-(2a_2^2+24a_2a_3+a_2a_0-24a_4a_2+6a_3a_0-12a_4a_0+24a_3^2)/K,$$
$$\beta_2=(12a_3^2+6a_2a_3-12a_4a_2+a_2^2)/K,$$
$$\alpha_0=(-12a_0a_4a_2-6a_0a_2a_3+12a_0a_3^2+a_0a_2^2+12a_2^3)/K,$$
$$\alpha_1=(24a_0a_4a_2+24a_0a_2a_3-24a_0a_3^2+10a_0a_2^2+a_2a_0^2-6a_3a_0^2-12a_4a_0^2-24a_2^3)/K,$$
$$\alpha_2=(12a_0a_4a_2+18a_0a_2a_3-12a_0a_3^2-a_0a_2^2+a_2a_0^2+6a_3a_0^2-12a_4a_0^2-12a_2^3)/K,$$
$$K=-6a_3a_0+12a_3^2+18a_2a_3+a_2a_0+13a_2^2-12a_4a_0-12a_4a_2.$$

Substituting obtained coefficients in last equation we obtain the residual of approximation:

$$-24a_0a_4a_2+144a_4^2a_0+144a_5a_2^2-288a_3a_4a_2+a_0a_2^2+144a_2a_3^2$$
$$+60a_3a_2^2-144a_4a_2^2+144a_3^3+12a_2^3-144a_5a_3a_0=0.$$

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